Online Appendix to "A Dodgson-Hare Synthesis,"

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Appendix Contents:

A1. Proofs of Propositions 1–3

Global Assumptions

Assume for simplicity that all voters have strict preferences among all candidates; and that both pairwise ties and plurality-score ties are broken by adding exogenous, heterogenous values (bounded between zero and one) to each candidate's score.

Proposition 1

Statement: If there is a sincere Condorcet winner X, and voters who prefer some other candidate Y change their votes to create a Smith set that includes Y, that Smith set must also still include X.

Proof:

1. Since X is the sincere Condorcet winner, a majority of voters sincerely prefer X to Y.

2. If only those who prefer Y to X vote strategically, X will still pairwise-beat Y in the tally after strategy has been employed.

3. If Y is in the Smith set, and X pairwise-beats Y, then X must also be in the Smith set. ■

Proposition 2

Statement: If sincere votes result in X being the Smith-Hare winner, no group of voters who prefer Y to X can make Y the Smith-Hare winner without including at least one member of the sincere Smith set in the observed Smith set.

Proof:

Case 1: Y is not in the sincere Smith set.

1.1. Since X is the sincere Smith-Hare winner, X is in the sincere Smith set.

1.2. Since Y is not in the sincere Smith set, 1.1 implies that a majority of voters prefer X to Y.

1.3. If only those who prefer Y to X vote strategically, X will still pairwise-beat Y in the tally after strategy has been employed.

1.4. If Y is in the Smith set, and X pairwise-beats Y, then X must also be in the Smith set.

Case 2: Y is in the sincere Smith set.

2.1. If Y is both the Smith-Hare winner and a member of the sincere Smith set, then Y is at least one candidate that is in both the sincere and observed Smith sets. ■

Proposition 3

Statement: In a one-dimensional spatial model, if voters who prefer Y to the sincere Condorcet winner X make Y the Condorcet-Hare winner by voting insincerely, there must exist some candidate or set of candidates who have the incentive to drop out in Step 2 of Dodgson-Hare, so that X still wins.

Proof:

1. Given a one-dimensional spatial model without voter indifference or unresolved pairwise ties, there must be a unique sincere Condorcet winner X in each election.

2. If votes are sincere, X wins both Condorcet-Hare and Dodgson-Hare.

3. Suppose that voters who prefer Y to X vote insincerely so that Y is the Condorcet-Hare winner. Without loss of generality, suppose that Y is to the west of X on a spectrum from west to east.

4. In order to make Y the Condorcet-Hare winner, the Y $\succ$ X voters must cause at least one other candidate Z to pairwise-beat X, so that X is not the Condorcet winner.

5. For a candidate Z to pairwise-beat X that does not pairwise-beat X given sincere voting, the strategic voters together with those who sincerely prefer Z to X must comprise a majority of the electorate. Therefore, any such candidate Z must be located to the east of X.

6. Since all candidates to the east of X prefer X to Y, they will be motivated to drop out if doing so changes the winner from Y to X. Thus, X will be the Dodgson-Hare winner. ■

A2. Comparing Dodgson's Pamphlets:[1] *Discussion*, *Suggestions*, and *Method*

A2.1. *A Discussion of the Various Methods of Procedure in Conducting Elections* (1873)

In this pamphlet (to be abbreviated as *Discussion*), Dodgson first introduced and critiqued six election rules: plurality (whoever gets the most votes wins), majority (elect any candidate if more than half vote in favor), pairwise elimination (with options considered in an arbitrary order, and defeated options not allowed to be considered a second time), Hare (in the form of a multi-round runoff, where the plurality loser is eliminated in each round), single-winner cumulative voting (which he calls "the method of marks"), and a procedure of nomination and voting in which nominated candidates who fail to get a majority cannot be considered again.

[1] As printed in Black (1958).

Next, Dodgson proposed a modified version of Borda in which "no election" may be included as a rankable option, and in which candidates ranked equally by a voter are awarded the number of points associated with the *highest* place they share on the ballot.[2] In adding this last feature, he joined the ranks of those concerned with Borda's vulnerability to strategic voting. Specifically, he worried that strategic voters would list their favorite candidate first, while listing all others as tied for last.

In *Discussion*, Dodgson gave an example in which *a* is the Borda winner although *b* has a narrow majority of first choice votes, and asserted: "There seems to be no doubt that *a* ought to be elected."

Per Black's (1958) examination of Dodgson's diary entries, Governing Body minutes, and a document with written vote totals from December 18, 1973 — the same day that Dodgson delivered *Discussion* — it seems that the Governing Body *initially* used Dodgson's modified Borda method in the course of selecting a reader on this day. However, after two candidates received similar scores, the Governing Body proceeded to a majority vote between the two, in which the candidate with the marginally lower initial score prevailed.[3]

A2.2. *Suggestions as to the Best Method of Taking Votes, Where More than Two Issues Are to Be Voted On* (1874)

By the time he wrote this second pamphlet (to be abbreviated as *Suggestions*; also, the shortest of the three), Dodgson had changed his mind from preferring Borda's approximately

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[2] For example, in a four-candidate election with candidates A through D, a ballot cast as A > B = C > D might count as four points for A, three points for B *and* C, and zero points for D.

[3] Thus, the actual method used on this occasion may have amounted to a Borda-runoff procedure, which is one of nine runoff voting rules considered in depth by Green-Armytage and Tideman (2020).

utilitarian approach to Condorcet's pairwise-majoritarian approach (despite not necessarily being aware of the names now associated with these concepts). In a short preface, he noted this about-face by writing "I do not now advocate the method, there proposed, as a good one to *begin* with. When other means have failed, it may prove useful, but that is not likely to happen often, and, when the difficulty does arise, the question what should next be done may fairly be debated on its own merits." He did not mention the idea of completing Condorcet with Borda again in this pamphlet or the next (despite substantial further discussion of Condorcet completion rules), but nonetheless this text anticipates what we now know as the Black rule.[4]

The main body of *Suggestions* outlines a procedure that may be used to identify the Condorcet winner when one exists, and map out the Smith set otherwise. In the initial stage, each voter indicates one candidate, and if any candidate receives a majority, they are elected. If no majority winner emerges, a sequence of pairwise votes begins with a contest between any two arbitrary candidates. The winner of this contest is then put up against some other arbitrary alternative, and the winner of this contest against yet another. This sequence reaches a conclusion if it establishes that one candidate defeats all others pairwise; if so, this Condorcet winner is elected.

Otherwise, it may reveal a majority rule cycle. Of this, Dodgson wrote: "If no settlement has been arrived at by § 3 or § 4, it will at least prove that the matter is one on which the meeting is *very evenly divided in opinion*. Such a state of things is of course very difficult to deal with, but the difficulty, though possibly not diminished, will certainly not have been increased by adopting the process I have here suggested." In other words, Dodgson here urged that the

[4] This name also derives from Black (1958), though in this case it is Part I (theory) rather than Part II (history), and specifically the chapter entitled "Which Candidate *ought* to be Elected?"

Condorcet winner should be selected when one exists; and suggested that otherwise, there is no obviously "correct" resolution.

A2.3. *A Method of Taking Votes on More than Two Issues* (1876)

In this third and last pamphlet on the subject of single-winner elections (to be abbreviated as *Method*), Dodgson retained his goal from *Suggestions* of finding the Condorcet winner when one exists, but refined his procedure for doing so in important ways. For example, as in *Suggestions*, his procedure in *Method* begins by having each voter indicate one candidate, and electing the majority winner if one exists. Both procedures proceed by searching for the Condorcet winner otherwise. However, while the previous proposal does so by voting on candidates two at a time, the updated proposal adopts the more convenient method of using ranked ballots.

The remainder of Dodgson's *Method* is discussed in the paper's main body.

A3. Description of Strategic Voting Algorithms Used

All of these algorithms proceed by first finding the candidate who wins with sincere voting; denote this candidate as w. Then they loop through the remaining candidates in search of a candidate q who can win the election if those who prefer q to w vote strategically.

It is convenient to define some additional notation at the outset. Let c, x, y, and z be candidate indices. Let v be a voter index. Let a 'strategic voter' be a voter who prefers q to w, and let a 'nonstrategic voter' be a voter who doesn't prefer q to w. Let V be the number of voters, let s_v be equal to one if and only if voter v is strategic, and let $S = \sum_{v=1}^{V} s_v$ be the number of strategic voters.

Let σ be a vector of positional scores, so that σ_c is the positional score of candidate c. As in the paper, let p denote the value of a second-choice vote (in a rule with a positional component), with the understanding that a first-choice vote is worth one point and a third-choice vote is worth zero points.

Let an overbar denote a variable including only the votes of nonstrategic voters, and a tilde denote a variable including only the votes of strategic voters; for example, let $\bar{\sigma}$ be the vector of positional scores summed only from the ballots of nonstrategic voters, and let $\tilde{\sigma}$ be the vector of positional scores summed only from the ballots of strategic voters.

A3.1. Positional Rules: Plurality, Borda, and Anti-Plurality

Positional rules are monotonic, so the only rankings that need to be considered are those that rank q first. Thus, calling the other candidates x and y, we only need to search over combinations of the orderings $q \succ x \succ y$ and $q \succ y \succ x$. An efficient way to proceed is to arbitrarily choose one candidate other than q to be denoted as y, and then to aim to give y as close a score to q as possible without going over. Then, we check to see whether this approach succeeds in electing q.

Thus, we want to maximize the number of $q \succ y \succ x$ votes that strategists cast, subject to the constraint that $\sigma_q > \sigma_y$. To find this maximum, define $\hat{\sigma}$ as the provisional vector of scores that arise if nonstrategic voters voted sincerely, and if strategic voters were able to indicate only a first preference for q, giving no points to the remaining candidates. That is,

$$\hat{\sigma}_c = \begin{cases} \bar{\sigma}_c + S & \text{if } c = q \\ \bar{\sigma}_c & \text{otherwise} \end{cases}$$

Defining κ as the number of $q \succ y \succ x$ votes that strategists cast, the $\sigma_q > \sigma_y$ constraint can be re-written as

$$\hat{\sigma}_q > \hat{\sigma}_y + \kappa p$$

$$\kappa < \frac{\hat{\sigma}_q - \hat{\sigma}_y}{p}$$

So, we choose the largest whole value of $\kappa \in [0, S]$ that satisfies this constraint, if such a value exists. Then, strategists cast κ votes in this manner and the remaining $S - \kappa$ votes as $q \succ x \succ y$.

A3.2. Elimination Rules: Hare, Baldwin, and Coombs

Define x as the candidate whom strategists aim to eliminate in the first round, and y as the candidate whom they aim to eliminate in the second round, leaving q as the winner. There is no simple way to know which candidate should be treated as x and which candidate should be y, so we loop over the two possible elimination orders that elect q.

In this case, the two orderings for strategists to consider are $q \succ y \succ x$ and $y \succ q \succ x$, because ranking x last maximizes the chance of eliminating x in the first round, without reducing q's chances of defeating y in the second round. Since $q \succ y \succ x$ votes are preferable for the purposes of the second round, we want to maximize the number of these votes, subject to the constraint that x has a lower positional score than y in the first round.

Thus, our constraint is $\sigma_y > \sigma_x$, which we can re-write as $\bar{\sigma}_y + \tilde{\sigma}_y > \bar{\sigma}_x + \tilde{\sigma}_x$, or

$$\tilde{\sigma}_y - \tilde{\sigma}_x > \bar{\sigma}_x - \bar{\sigma}_y$$

Since all strategists will rank x last, $\tilde{\sigma}_x = 0$. Defining ψ as the number of strategic $y \succ q \succ x$ votes, and $\kappa = S - \psi$ again as the number of strategic $q \succ y \succ x$ votes, we find that $\tilde{\sigma}_y = \psi + \kappa p$, or $\tilde{\sigma}_y = \psi + Sp - \psi p$. Thus, we can re-write our constraint as

$$\psi + Sp - \psi p > \bar{\sigma}_x - \bar{\sigma}_y$$

$$\psi > \frac{\bar{\sigma}_x - \bar{\sigma}_y - Sp}{1 - p}$$

So, we choose the smallest whole value of $\psi \in [0, S]$ that satisfies this constraint, if such a value exists. Then we cast the strategic votes accordingly, and see if q is the winner.

A3.3. Condorcet-Positional Rules: Condorcet-Plurality, Black, and Condorcet-Anti-Plurality

This algorithm proceeds much as the positional rules algorithm does, except for two modest differences. First, this program also checks whether strategic voters can make q the Condorcet winner, by all ranking him first. (This will only work in cases without a sincere Condorcet winner.) Second, this program tries both possible assignments of the designations x and y to the two candidates other than q. Other than that, the principle is the same: strategists cast a mix of $q \succ y \succ x$ votes and $q \succ x \succ y$ votes, but they maximize the former and minimize the latter subject to the constraint that $\sigma_q > \sigma_y$.

A3.4. Condorcet-Elimination Rules: Condorcet-Hare and Condorcet-Coombs

This algorithm is almost the same as the elimination rules algorithm, except that it also checks whether q becomes the Condorcet winner if all strategic voters rank q first.

A3.5. Range and Approval

The range voting algorithm tests whether q wins if all strategic voters give q the maximum rating, and all other candidates the minimum rating.

The approval algorithm tests whether q wins if all strategic voters approve only candidate q.

A3.6. Minimax

If w is a sincere Condorcet winner, or if there is a sincere $w \to q \to z \to w$ cycle, strategists all cast the ordering $q \succ z \succ w$. If there is a sincere $w \to z \to q \to w$ cycle, they cast the ordering $q \succ w \succ z$.

A3.7. Dodgson-Hare

The algorithm for detecting strategic possibility in Dodgson-Hare *without* the cooperative victim strategy is the same as Condorcet-Hare, with the added condition that candidate y must at least weakly prefer candidate q to candidate x; as otherwise, y will drop out and the strategists' attempt to elect q will fail.

The algorithm for detecting strategic possibility in Dodgson-Hare *with* the cooperative victim strategy available first checks whether there is a sincere Condorcet winner. If there is none, or if strategy is deemed to be possible without the cooperative victim strategy as described above, then the election is manipulable. Otherwise, for each potential strategic candidate $q \neq w$, the algorithm requires the existence of a candidate $y \neq q$ such that candidate w prefers q to y. If such a candidate is found, the algorithm searches for an allocation of strategic votes such that q pairwise-beats y and y pairwise-beats w (thus creating a $w \rightarrow q \rightarrow y \rightarrow w$ cycle), and such that q is the plurality loser — so that if there are no withdrawals, q will be eliminated first, w will be eliminated second, and y will win. If such an allocation is possible, then w may theoretically be induced to withdraw in favor of q, to prevent the election of y.

Appendix References

Black, Duncan (1958) *The Theory of Committees and Elections.* Cambridge University Press.

Green-Armytage, James, and T. Nicolaus Tideman (2020) Selecting the Runoff Pair." *Public Choice* 182:1, 119-137.