Worked problem on quantity-based price discrimination

Setup

Suppose that a monopolist sells a good with zero marginal cost of production, and faces two types of consumers, type A and type B, whose inverse demand curves are

$$p^A(q^A) = 12 - \frac{1}{2}q^A \qquad\qquad p^B(q^B) = 8 - \frac{1}{2}q^B$$

Here, q^A and q^B represent the quantity of the good that a consumer (of type A and type B, respectively) receives. For simplicity, suppose a consumer has equal likelihood of being type A or type B.

Part 1: Perfect discrimination

In part 1, suppose that the firm can tell whether each consumer is of type A or of type B. The firm can maximize its profit using a two-part pricing scheme in which, rather than paying per unit, each consumer pays a fee in exchange for the purchase of a fixed quantity determined by the firm. Specifically, the firm should charge each consumer a fee equal to the total area under his or her demand curve, in exchange for the quantity at which the value of the inverse demand function is zero.

That is, for consumers of type A, the firm should offer a quantity of $q^A = 24$ in exchange for a fee of $F^A = (1/2)(24)(12) = 144$. For consumers of type B, the firm should offer a quantity of $q^B = 16$ in exchange for a fee of $F^B = (1/2)(16)(8) = 64$. Each consumer must either take or leave the offer as it is; they may not purchase or sell on a per-unit basis. For simplicity, we assume that the consumers accept the offer by default if they are just indifferent, as they are in this case.

Since the two types of consumers are equally common, we can without loss of generality define the firm's producer surplus in terms of the producer surplus from a representative pair of consumers. Because marginal costs are zero, this is equivalent to the sum of the fees received from each type of consumer. So, in this case, we calculate producer surplus as $\pi = F^A + F^B = 144 + 64 = 208$.

Note that, although consumer surplus is zero, the outcome is Pareto efficient. The firm has effectively captured all potential surplus that the market can generate, and the number of units received by each consumer is equivalent to the number that they would demand if the per-unit price were set equal to marginal cost.

On the other hand, if the firm charges each group a positive per-unit price instead of a fee in exchange for a fixed quantity, even the best possible price will fall far short of both profit maximization and overall efficiency. For example, setting marginal revenue of an additional unit for a consumer of type A equal to the marginal cost would give $12 - q^A = 0$, $q^A = 12$, $p^A = 6$, consumer surplus $\kappa^A = (1/2)(12)(6) = 36$, producer surplus $\pi^A = (12)(6) = 72$, total surplus $\sigma^A = 108$, and deadweight loss $\delta^A = (1/2)(12)(6) = 36$. Applying the same method to consumers of type B gives quantity $q^B = 8$, $p^B = 4$, consumer surplus $\kappa^A = (1/2)(8)(4) = 16$, producer surplus $\pi^B = (8)(4) = 32$, total surplus $\sigma^B = 48$, and deadweight loss $\delta^B = (1/2)(8)(4) = 16$. Total surplus from the two consumer types combined is $\sigma^A + \sigma^B = 108 + 48 = 156$, which falls short of the maximum surplus value of 208 by the combined deadweight loss of $\delta^A + \delta^B = 36 + 16 = 52$.

Part 2: Imperfect discrimination

For the rest of the exercise, suppose that the firm *cannot* distinguish between the type A consumers and the type B consumers. To be clear, the firm knows the demand curve of the two types, and is aware that each type is equally common, but it is unable to determine the type of any one particular consumer. It is still optimal for the firm to offer two distinct packages with fixed quantities rather than to sell the good on a per-unit basis, but it can no longer prevent a type A consumer from buying the package that was meant for the type B consumers, or vice versa.

We consider this problem in three parts. First, we consider how much profit the firm can make if it offers packages with the quantities that would be demanded by the two types of consumers given marginal (zero) cost pricing. Second, we show that a reduction in the smaller package can increase the firm's profits, by increasing the fee that the high-demand consumers will be willing to pay for the larger package. Third, we find the optimal size of the smaller package, and thus the firm's profit-maximizing pricing scheme.

2.1: Package deals with efficient quantities

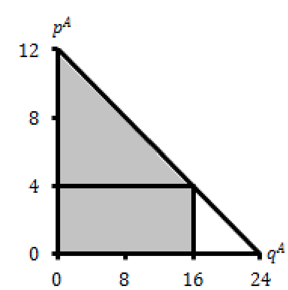
Since the firm offers packages of $q = 24$ and $q = 16$ in the case with perfect price discrimination, it is reasonable to first determine their profit if they offer the same packages in this case as well. To do this, we must find the profit-maximizing fees for these packages.

Define V_q^A and V_q^B as the values of a package with quantity q to a consumer of type A and a consumer of type B, respectively. Define F_q as the fee charged for a package with quantity q.

The value to a type B consumer of a package with $q = 16$ units is $V_{16}^B = (1/2)(16)(8) = 64$. This is also the profit-maximizing fee for the smaller package: If the firm charges more, the low-demand consumers will drop out of the market. If the firm charges less, they will extract less revenue from the low demanders, and they will also have to offer a lower fee to the high demanders to prevent them from choosing the smaller package. So we have $F_{16} = V_{16}^B = 64$.

Determining the optimal fee for the larger package requires an extra step. To see this, consider first what would happen if the firm charged $F_{24} = V_{24}^A = 144$. Whereas a consumer of type A would receive a surplus of $\kappa^A = V_{24}^A - F_{24} = 0$ if she bought the larger package, she would receive a surplus of $\kappa^A = V_{16}^A - F_{16}$ if she bought the smaller package.

We can evaluate V_{16}^A as the area under a type-A consumer's demand curve up to a quantity of $q = 16$, i.e. the shaded area in the graph to the right. Thus, $V_{16}^A = (1/2)(16)(8) + (16)(4) = 128$. So a high-demand consumer gets surplus of $\kappa^A = 128 - 64 = 64$ from buying the smaller package, which he will therefore choose to do. This means that no one will be willing to buy the larger package if $F_{24} = 144$ and $F_{16} = 64$.



A high-demand consumer will only be willing buy the larger package if

$$V_{24}^A - F_{24} \geq V_{16}^A - F_{16}$$
$$F_{24} \leq V_{24}^A - V_{16}^A + F_{16}$$

For simplicity, suppose that a consumer will choose the larger package if they are precisely indifferent between the two packages. With this assumed, $F_{24} = V_{24}^A - V_{16}^A + F_{16}$ gives the highest price that the firm can sell the larger package for. $V_{24}^A = 144$, $V_{16}^A = 128$, and the profit-maximizing fee for the 16-unit package is $F_{16} = 64$, so the profit-maximizing price of the 24-unit package is $F_{24} = 80$, and the highest possible producer surplus given packages of $q = 24$ and $q = 16$ is $\pi = F_{24} + F_{16} = 80 + 64 = 144$.

2.2. Reducing the smaller package can increase profits

Next, consider what prices the firm can charge if it keeps the larger package $q = 24$, but reduces the smaller package to $q = 10$. For the reasons cited above, it is optimal to charge a fee of V_{10}^B for the smaller package, so we have $F_{10} = V_{10}^B = (1/2)(10)(5) + (10)(3) = 55$.

What fee can the firm induce the high demanders to pay for the larger package? The incentive compatibility constraint is

$$V_{24}^A - F_{24} \geq V_{10}^A - F_{10}$$

The value of the 10-pack to the high demander is $V_{10}^A = (1/2)(10)(5) + (10)(7) = 95$, so we have

$$144 - F_{24} \geq 95 - 55$$

So the profit-maximizing fees are $F_{24} = 104$ and $F_{10} = 55$, which give profit $\pi = F_{24} + F_{10} = 159$.

Two things are worth noting here. First, the firm's profit has increased; although they receive less revenue from the low demanders as a result of offering a smaller package, the greater revenue received from the high demanders more than makes up for this. The second thing is that whereas the situation with packages of $q = 24$ and $q = 16$ is Pareto efficient, this case is not. That is, in the process of extracting more revenue from the high demanders, the firm has introduced deadweight loss into their transaction with the low demanders.

2.3. Finding the profit-maximizing size of the smaller package

We have found that decreasing the size of the smaller package can increase the firm's profit. But up to what point does this hold? Now we want to complete the problem by determining the profit-maximizing size of the two packages, and the fees that go along with this. Our strategy will be to express the firm's profit as a function of the size of the smaller package, which we will call x, and then to set the first derivative of this function equal to zero.

We've defined profit as $\pi = F_{24} + F_x$, so we want to evaluate both of these terms. We know that F_x should be equal to the value of the low demander for the smaller package, i.e. that $F_x = V_x^B$. Also, we know that F_{24} is constrained by the weak inequality $V_{24}^A - F_{24} \geq V_x^A - F_x$, which becomes an equality when the firm is maximizing its profit. Putting these together, we have

$$\pi = [V_{24}^A - V_x^A + V_x^B] + [V_x^B]$$
$$\pi = V_{24}^A - V_x^A + 2V_x^B$$

Setting the first derivative of this profit function equal to zero, we have

$$\frac{d\pi}{dx} = -\frac{dV_x^A}{dx} + 2\frac{dV_x^B}{dx} \overset{\text{set}}{=} 0$$
$$\frac{dV_x^A}{dx} = 2\frac{dV_x^B}{dx}$$

What remains is to evaluate these derivatives. Luckily, this is not so hard: they are simply equal to the values of inverse demand functions of the two consumers, at a quantity of x. That is, a consumer's willingness to pay for the next unit on the margin also gives us the rate at which his overall benefit from the good (in dollar terms) is increasing with quantity. Therefore, we have

$$p^A(x) = 2p^B(x)$$
$$12 - \frac{1}{2}x = 2\left(8 - \frac{1}{2}x\right)$$
$$x = 8$$

So the firm maximizes profit by offering a smaller package of 8, and a larger package of 24. From here, the optimal fees and thus the maximum profit are straightforward to calculate.

$$F_8 = V_8^B = \left(\frac{1}{2}\right)(8)(4) + (8)(4) = 48$$
$$F_{24} = V_{24}^A - V_x^A + F_8 = 112$$
$$\pi = F_{24} + F_8 = 160$$