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Optimal government size in a simple model

In this simple model, there are two types of markets: private goods markets, which are efficient in the absence of taxes, and public goods markets, in which supply depends entirely on government expenditure. It is assumed that governments can only get revenue to finance this supply by levying per-unit taxes on the private goods markets; thus, the challenge is to optimally balance the efficiency losses associated with this taxation against the efficiency gains associated with providing public goods.

In order to facilitate this balancing act, define λ as the shadow value cost of spending the last tax dollar, and define μ as the shadow cost of raising the last tax dollar. Given these definitions and the assumption of a balanced budget constraint, it should be intuitive that at the optimal government size, $\lambda = \mu$.

In part 1, I investigate the optimal level of a public good's provision, given a particular value of μ. In part 2, I investigate the optimal level of taxation on a the market for a private good, given a particular value of λ. In part 3, I discuss how these two analyses can be combined to find the optimal government size. In part 4, I give a numerical example.

Part 1: Spending money on public goods

There are several ways to model the benefits of government spending, but for simplicity I choose to focus on a case in which goods are either purely private or purely public, and in which the public goods are not provided at all in the absence of government expenditure. Further, the government's budget must balance; i.e. it cannot finance spending through debt, and it has no benefits from saving. The quantity of the public good is represented by y, and the marginal benefits and marginal costs of provisions are given by the linear functions

$$MB(y) = \mathrm{A} - \mathrm{B}y \qquad\qquad MC(y) = \Gamma + \Delta y$$

Here, A, B, Γ, and Δ are non-negative parameters, such that $\mathrm{A} - \Gamma > 0$ and $\mathrm{B} + \Delta > 0$.

We have provided an optimal amount of the public good when λ, the social benefit from the last dollar spent on the public good is just equal to μ the social cost of raising the last dollar of tax revenue. To proceed, let's assume a particular value of μ, and calculate the optimal value of y, at which $\lambda(y) = \mu$.

The shadow value of the last dollar of spending, λ, is the ratio of the marginal benefit of the public good to the marginal cost of the public good, i.e.

$$\lambda(y) = \frac{MB(y)}{MC(y)}$$

This shadow value $\lambda(y)$ declines as y increases. Setting it equal to the assumed shadow cost μ:

$$\frac{MB(y)}{MC(y)} = \mu$$

$$MB(y) = \mu MC(y)$$

This gives the optimal value of y, given a particular value of μ.

Putting this equation in terms of the parameters of our linear marginal benefit and cost functions, we have

$$\mathrm{A} - \mathrm{B}y = \mu(\Gamma + \Delta y)$$

$$\boxed{y^o(\mu) = \frac{\mathrm{A} - \mu\Gamma}{\mathrm{B} + \mu\Delta}}$$

Given any value of y, we can find the necessary expenditure (total cost) by taking the anti-derivative of the marginal cost function. (We can also find this graphically by adding a rectangle to a right triangle.)

$$\boxed{E(y) = \Gamma y + \frac{1}{2}\Delta y^2}$$

Part 2: Raising revenue in private, initially-efficient markets

Let x be the quantity of a private good, for which there is a competitive, efficient market. Assume that aggregate marginal benefit and marginal cost are given by the following linear functions

$$MB(x) = \alpha - \beta x \qquad\qquad MC(x) = \gamma + \delta x$$

Let τ be a per-unit tax on the good. In the equilibrium with the tax, $MB(x) = MC(x) + \tau$, so

$$\alpha - \beta x = \gamma + \delta x + \tau$$

$$\alpha - \gamma - \tau = x(\beta + \delta)$$

$$x(\tau) = \frac{\alpha - \gamma - \tau}{\beta + \delta}$$

$x(\tau)$ gives the equilibrium quantity of the good, conditional on τ. Since the market is initially efficient, the surplus-maximizing quantity x^o can be derived by simply setting $\tau = 0$:

$$x^o = \frac{\alpha - \gamma}{\beta + \delta}$$

Let $R(\tau) = \tau x^*(\tau)$ be the government's revenue from the tax:

$$\boxed{R(\tau) = \frac{(\alpha - \gamma)\tau - \tau^2}{\beta + \delta}}$$

$$R'(\tau) = \frac{\alpha - \gamma - 2\tau}{\beta + \delta}$$

It's important to note that $R(\tau)$ is concave, i.e. that $R''(\tau) < 0$.

Let $DWL(\tau)$ be the deadweight loss (or loss in surplus) as a result of the tax. The formal definition uses an integral, but because supply and demand are linear, deadweight loss can be calculated as the area of a triangle.

$$DWL(\tau) = \int_{x(\tau)}^{x^o} MB(x) - MC(x)\, dx$$

$$DWL(\tau) = \frac{1}{2}\tau\big(x^o - x(\tau)\big) = \frac{1}{2}\tau\left(\frac{\tau}{\beta + \delta}\right)$$

$$DWL(\tau) = \frac{\tau^2}{2(\beta + \delta)}$$

$$DWL'(\tau) = \frac{\tau}{\beta + \delta}$$

It's important to note that $DWL(\tau)$ is convex, i.e. that $DWL''(\tau) > 0$. When τ is close to zero, the marginal deadweight loss is close to zero as well, but as τ increases, the ratio of deadweight loss to revenue grows continuously.

Let λ be the shadow value of government spending; that is, assume that an additional dollar of public spending (e.g. on public goods) creates $\$\lambda$ in public consumption value. Taking a particular value of λ as given, we want to tax up to the point that μ, the shadow cost of the last tax dollar, is just equal to λ.

What is the social cost of the last dollar of tax revenue raised? It is the rate at which revenue plus deadweight loss increases as revenue increases, i.e.

$$\mu(\tau) = \frac{R'(\tau) + DWL'(\tau)}{R'(\tau)} = 1 + \frac{DWL'(\tau)}{R'(\tau)}$$

Setting $\mu(\tau) = \lambda$, we have

$$\frac{R'(\tau) + DWL'(\tau)}{R'(\tau)} = \lambda$$

$$\lambda R'(\tau) = R'(\tau) + DWL'(\tau)$$

$$(\lambda - 1)R'(\tau) = DWL'(\tau)$$

$$(\lambda - 1)\frac{\alpha - \gamma - 2\tau}{\delta + \beta} = \frac{\tau}{\delta + \beta}$$

$$(\lambda - 1)(\alpha - \gamma) = (2\lambda - 1)\tau$$

$$\boxed{\tau^o = \frac{\lambda - 1}{2\lambda - 1}(\alpha - \gamma)}$$

Thus, $\lambda > 1$ implies $\tau^o > 0$. That is, if the shadow value of expenditure is greater than one, it's optimal to have some positive tax rate despite the fact that this creates deadweight loss in the market for the private good. As one would expect, τ^o depends positively on λ, and positively on $\alpha - \gamma$, i.e. the maximum difference between marginal benefit and marginal cost.

Part 3: Global optimality

So far, we've derived conditions and expressions for the optimal amount of spending as a function of the shadow cost of tax revenue, and the optimal amount of taxation as a function of the shadow value of expenditure. Now, we consider how this balance may be established.

If we begin with no taxation and no spending, the shadow cost of raising a tax dollar, $\mu(\tau)$, is initially one:

$$\mu(\tau) = \frac{R'(\tau) + DWL'(\tau)}{R'(\tau)} = 1 + \frac{DWL'(\tau)}{R'(\tau)}$$

$$R'(\tau) = \frac{\alpha - \gamma - 2\tau}{\beta + \delta} \qquad\qquad DWL'(\tau) = \frac{\tau}{\beta + \delta}$$

$$\mu(0) = 1$$

However, as the tax rate increases, the shadow cost of the last tax dollar increases as well; this is made clear by the above equations because higher values of τ lead to both higher values of $DWL'(\tau)$ and lower values of $R'(\tau)$.

On the other hand, when there is no taxation or spending, the shadow value of spending a tax dollar, $\lambda(y)$, is greater than one:

$$\lambda(y) = \frac{MB(y)}{MC(y)} = \frac{\mathrm{A} - \mathrm{B}y}{\Gamma + \Delta y}$$

$$\lambda(0) = \frac{\mathrm{A}}{\Gamma} > 1$$

Furthermore, whereas the shadow cost $\mu(\tau)$ is increasing in τ, the shadow value $\lambda(y)$ is decreasing in y, because $MB(y)$ decreases with y while $MC(y)$ decreases with y.

Thus, when government size is zero, the shadow value of spending is greater than the shadow cost of taxation – i.e. $\lambda > \mu$ – which indicates that government is 'too small', i.e. less than its optimal size, because raising and spending an additional dollar of tax revenue would generate more social benefit than social cost. As we increase the size of government, the shadow value λ falls continuously, while the shadow cost μ increases continuously, until they reach a unique common value, which we can denote as σ. If we continue increasing the size of government to a point at which $\mu > \lambda$, we have a government that is 'too big', i.e. greater than its optimal size, because raising and spending the last tax dollar generated more social cost than social benefit. Therefore, government size is precisely optimal when the shadow benefit of spending is just equal to the shadow cost of taxation, i.e. when $\lambda(y) = \mu(\tau) \equiv \sigma$.

Part 4: Example

Suppose that the marginal benefit and marginal cost for the private good are given by

$$MB(x) = 100 - 2x \qquad\qquad MC(x) = 10 + x$$

Suppose also that the marginal benefit and marginal cost for the public good are given by

$$MB(y) = 100 - 2y \qquad\qquad MC(y) = 20$$

First, consider the case in which $\tau = 0$, and thus $y = 0$. The private goods market reaches equilibrium where $100 - 2x = 10 + x$: this implies that $x = 30$, and $p = 40$. Consumer surplus is $CS_x = (1/2)(30)(60) = 900$, producer surplus is $PS_x = (1/2)(30)(30) = 450$, and so the total economic surplus from this market is $TES_x = CS_x + PS_x = 1350$. Since zero units of the public good are provided, that market generates a total economic surplus of zero: $TES_y = 0$. Thus, the total economic surplus from both markets combined is $TES_x + TES_y = 1350$.

Given $\tau = 0$ and therefore $y = 0$, we can calculate the values of μ and λ as follows:

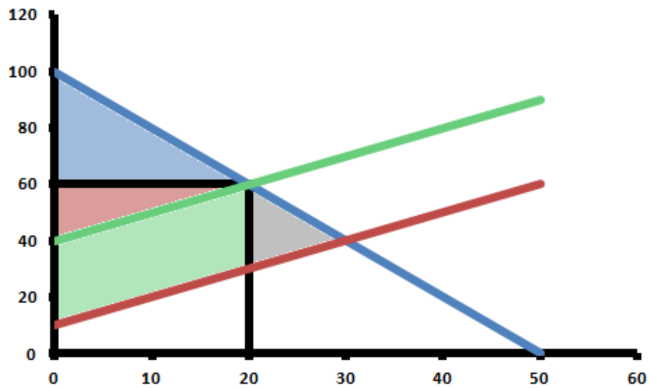
$$\mu(\tau) = 1 + \frac{DWL'(\tau)}{R'(\tau)} = 1 + \frac{\left(\frac{1}{3}\tau\right)}{\left(30 - \frac{2}{3}\tau\right)} \qquad\qquad \mu(0) = 1$$

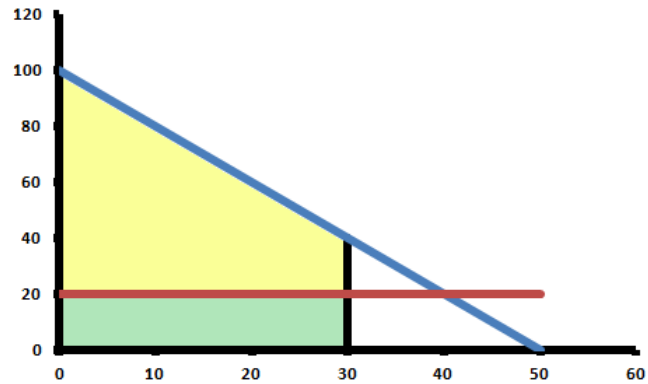
$$\lambda(y) = \frac{MB(y)}{MC(y)} = \frac{100 - 2y}{20} \qquad\qquad \lambda(0) = 5$$

Since the shadow value of spending is five times greater than the shadow cost of raising revenue, it is logical to increase the size of government.

Next, let's consider the case in which $\tau = 30$, with a view toward verifying that this is in fact the optimal level of taxation. In the private goods market, given $\tau = 30$, we have $x = 20$, $p = 60$, $R = 600$, $CS_x = 400$, $PS_x = 200$, and $TES_x = 600$. The revenue $R = 600$ allows us to provide $y = 30$ units of the public good, resulting in $TES_y = CS_y = (1/2)(30)(60) +$

$(30)(40) = 2100$. Therefore, the total economic surplus generated by both markets is $TES_x + TES_y = 2700$. The state of both markets given $\tau = 30$ is depicted below.





We've improved from the surplus of 1350 that results from $\tau = 0$. However, to verify that no further improvement is possible, we want to compare the shadow cost μ to the shadow benefit λ:

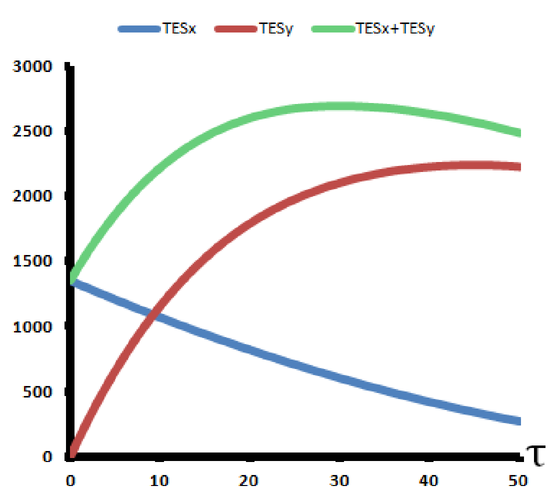
$$\mu(30) = 1 + \frac{DWL'(30)}{R'(30)} = 1 + \frac{\left(\frac{1}{3}(30)\right)}{\left(30 - \frac{2}{3}(30)\right)} = 1 + 1 = 2$$

$$\lambda(30) = \frac{MB(30)}{MC(30)} = \frac{100 - 2(30)}{20} = \frac{40}{20} = 2$$

Since $\mu = \lambda$ when $\tau = 30$ and therefore $y = 30$, government is optimally sized.

For example, if we were to increase τ still further, let's say to $\tau = 36$, we have $x = 18$, $p = 64$, $R = 648$, $CS_x = 324$, $PS_x = 162$, $TES_x = 486$, $y = 32.4$, $TES_y = CS_y = 2190.2$, and $TES_x + TES_y = 2676.2$, which is less than the surplus of 2700 generated by $\tau = 30$. This is a government that is 'too big'.

The graph to the right shows TES_x, TES_y, and $TES_x + TES_y$, all as a function of τ: our objective is to maximize $TES_x + TES_y$, which happens at $\tau = 30$.



One method of finding the optimal government size is to guess and iterate on possible values of σ, checking each time to see whether the government's budget is balanced when they choose the tax τ so that $\mu(\tau) = \sigma$, and the public good quantity y so that $\lambda(y) = \sigma$. Doing this with pen and paper (as opposed to a computer) can be tedious, but if one is given the correct value of σ, it's relatively straightforward and instructive to complete the rest of the process, i.e. to (1) find the optimal tax $\tau^o(\sigma)$ and the resulting revenue $R\big(\tau^o(\sigma)\big)$, (2) find the optimal output $y^o(\sigma)$ and the resulting expenditure $E\big(y^o(\sigma)\big)$, and (3) verify that the budget is balanced, i.e. that $R\big(\tau^o(\sigma)\big) = E\big(y^o(\sigma)\big)$.

For example, suppose that we were given only the following information:

$$MB(x) = 100 - 2x \qquad MC(x) = 10 + x \qquad MB(y) = 100 - 2y \qquad MC(y) = 20 \qquad \sigma = 2$$

First, we can find the optimal quantity of the public good given the assumption that $\mu = 2$:

$$MB(y) = \mu MC(y) \qquad\qquad 100 - 2y = (2)20$$

$$y^o = 30 \qquad\qquad E(y^o) = 600$$

Second, we can find the optimal tax given the assumption that $\lambda = 2$:

$$100 - 2x = 10 + x + \tau \qquad\qquad x(\tau) = 30 - \tfrac{1}{3}\tau$$

$$R(\tau) = 30\tau - \tfrac{2}{3}\tau^2 \qquad\qquad R'(\tau) = 30 - \tfrac{1}{3}\tau$$

$$DWL(\tau) = \tfrac{1}{6}\tau^2 \qquad\qquad DWL'(\tau) = \tfrac{1}{3}\tau$$

$$\lambda R'(\tau) = R'(\tau) - DWL'(\tau) \qquad\qquad (\lambda - 1)R'(\tau) = DWL'(\tau)$$

$$(1)\left(30 - \tfrac{1}{3}\tau\right) = \tfrac{1}{3}\tau \qquad\qquad \tau^o = 30$$

$$x(\tau^o) = 20 \qquad\qquad R(\tau^o) = 600$$

Finally, we verify that $E(y^o) = R(\tau^o)$, which demonstrates that these two assumptions of $\lambda = 2$ and $\mu = 2$ are consistent with one another. Thus, we've verified that $\sigma = 2$ is both the shadow value of spending and the shadow cost of taxation, when the government is optimally-sized.