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Two simple proofs of the full insurance theorem

Definitions

Let $\pi \in (0,1)$ be the probability that an injury will occur. Let $m \in (0,1)$ be the insurance premium rate, and let x be the number of insurance shares that the person buys. That is, if he buys x shares, he pays a premium of mx regardless, but receives a payoff of x from the insurance company if the injury occurs.

Let w be the individual's starting wealth, let I be the raw cost of the injury, and let c_n be his consumption if injured, and c_i be his consumption if not injured. Thus we can write consumption in the two states of the world as

$$c_n = w - mx \qquad\qquad c_i = w - I + (1-m)x$$

Let $U = \pi V(c_i) + (1-\pi)V(c_n)$ be the individual's expected utility, where $V(c)$ is his Von Neumann-Morgenstern utility of consumption function. Assume $V'(c) > 0$ and $V''(c) < 0$, i.e. that utility is increasing in consumption, and that the individual is risk-averse.

Suppose that the individual's objective is to chose the value of x that maximizes his expected utility.

First proof

We want to show that if $m = \pi$ (i.e. if the insurance price is actuarially fair), that the individual will choose full insurance, so that $x = I$, and $c_i = c_n$.

Using the definitions of U, c_n, and c_i, expected utility can be written in terms of x as

$$U = \pi V(w - I + (1-m)x) + (1-\pi)V(w - mx)$$

Next, we set the derivative of U with respect to x equal to zero, to find the maximum. Note that we use the chain rule here.

$$\frac{dU}{dx} = \pi V'(w - I + (1-m)x)(1-m) + (1-\pi)V'(w - mx)(-m) \overset{\text{set}}{=} 0$$

Now that we've taken the derivative using the definitions of c_i and c_n, we can once again just write c_i and c_n in their place, to simplify the expression. Then, we have

$$(1-m)\pi V'(c_i) - m(1-\pi)V'(c_n) = 0$$

Rearranging a bit, we can first re-write this equation as

$$(1-m)\pi V'(c_i) = m(1-\pi)V'(c_n)$$

... and then, as

$$\boxed{\frac{V'(c_i)}{V'(c_n)} = \frac{\left(\frac{m}{1-m}\right)}{\left(\frac{\pi}{1-\pi}\right)}}$$

Now, we can see that if $m = \pi$, $V'(c_i) = V'(c_n)$, which implies that $c_i = c_n$, and thus $x = I$. Which is what we wanted to demonstrate.

Also, if $m > \pi$, which is the more common case in reality, this implies that $V'(c_i) > V'(c_n)$, and thus that $c_i < c_n$. That is, if the premium rate is higher than what is actuarially fair, individuals will choose incomplete insurance, so that their consumption will still be diminished if the adverse event occurs, even after insurance has been taken into account.

Second proof

Begin with the equations for consumption in the two states of the world, $c_n = w - mx$ and $c_i = (w - I) + (1 - m)x$. We would like to combine these into a single budget constraint with some linear expression of c_n and c_i on the left hand side, so we multiply the second equation by $m/(1 - m)$ to get

$$\frac{m}{1-m} c_i = \frac{m}{1-m}(w - I) + mx$$

Then, we can add this to the equation $c_n = w - mx$, so that the x terms will cancel:

$$c_n + \frac{m}{1-m} c_i = w + \frac{m}{1-m}(w - I)$$

Thus, we see that to get one unit of consumption in the state with no injury, we have to give up $m/(1 - m)$ units of consumption in the state with injury. So, we can say that the effective 'price ratio' between consumption in the two states is

$$\frac{p_{c_n}}{p_{c_i}} = \frac{1}{\left(\frac{m}{1-m}\right)} = \frac{1-m}{m}$$

Recall the individuals expected utility function, $U = \pi V(c_i) + (1 - \pi)V(c_n)$. The marginal expected utility of consumption in the two states can be given by

$$MU_{c_n} = (1-\pi)V'(c_n) \qquad\qquad MU_{c_i} = \pi V'(c_i)$$

Thus, the ratio of these marginal expected utilities is

$$\frac{MU_{c_n}}{MU_{c_i}} = \frac{(1-\pi)V'(c_n)}{\pi V'(c_i)}$$

Setting the 'price ratio' equal to the ratio of marginal expected utilities, we have

$$\frac{1-m}{m} = \frac{(1-\pi)V'(c_n)}{\pi V'(c_i)}$$

We can rearrange this to get the same equation as in the first proof, i.e.

$$\frac{V'(c_i)}{V'(c_n)} = \frac{\left(\frac{m}{1-m}\right)}{\left(\frac{\pi}{1-\pi}\right)}$$