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Strategic Voting and Nomination

Abstract: Using computer simulations based on three separate data generating processes, I estimate the fraction of elections in which sincere voting is a core equilibrium given each of eight single-winner voting rules. Additionally, I determine how often each rule is vulnerable to simple voting strategies such as 'burying' and 'compromising', and how often each rule gives an incentive for non-winning candidates to enter or leave races. I find that Hare is least vulnerable to strategic voting in general, whereas Borda, Coombs, approval, and range are most vulnerable. I find that plurality is most vulnerable to compromising and strategic exit (causing an unusually strong tendency toward two-party systems), and that Borda is most vulnerable to strategic entry. I use analytical proofs to provide further intuition for some of my key results.

1. Introduction

For many who seek to improve the political process, alternative voting rules offer the possibility of transformative change; however, there is no consensus on which rule is best. When evaluating these systems, we must consider the extent to which they will encourage strategic behavior. I distinguish between two basic types of election strategy: The first is strategic voting, which means voters reporting preferences that differ from their sincere appraisal of the candidates. The second is strategic nomination, which means non-winning candidates attempting to change the result by entering or exiting races.

Since Gibbard (1973) and Satterthwaite (1975) demonstrated that all practical single-winner election rules create incentives for strategic voting in at least some situations,[1] several authors have attempted to assess the degree to which different voting rules are susceptible to manipulation. There is no universally accepted way to measure this vulnerability, but one of the most common approaches has been to estimate the fraction of elections in which manipulation is logically possible, given some assumptions about the distribution function that governs voters' preferences over candidates. Some papers are concerned with the probability that an individual voter will be able to change the result to his own benefit by voting insincerely,[2] while others are concerned with the probability that a coalition of voters will be able to change the result to all of its members'

[1] Specifically, if there are more than two candidates for a single office, and a non-dictatorial election method allows voters to rank the candidates in any order, then there must be some profile of voter preferences under which at least one voter can get a preferred result by voting insincerely. This well-known 'Gibbard-Satterthwaite theorem' relies in turn on the even more well-known 'Arrow theorem'; for this, see Arrow (1951, rev. ed. 1963).

[2] For example, Nitzan (1985), Kelly (1993), Smith (1999), and Aleskerov and Kurbanov (1999). Saari (1990) focuses on 'micro manipulations', i.e. strategic incursions by groups of arbitrarily small size.

mutual benefit by voting insincerely,[3] and still others are concerned with both.[4] Here, I focus on the second of these, i.e. coalitional manipulation.

In this paper, I extend the strategic voting literature in at least four ways. First, I produce separate results for two distinct types of strategic voting – ‘compromising’ and ‘burying’ – which have different implications for political behavior, and examine the effects of limiting voters to a ‘simple’ strategy that combines these. Second, whereas most papers that give numerical estimates of voting rules’ vulnerability to coalitional manipulation are limited to a fixed number of candidates,[5] this paper presents algorithms that can generate estimates for any number of candidates. It is not practical to solve this problem using brute force, so I create a fundamentally distinct algorithm for each voting rule, based on the logical conditions that determine whether manipulation is possible. Third, whereas many papers in the literature have based their results on the assumption of a single data generating process,[6] I perform each of my strategic voting analyses given three different data sources: a spatial model, an impartial culture model, and survey responses from the American National Election Studies. As the latter derives from real preferences of citizens over politicians, it provides a useful complement to the more abstract models of voter preferences. By performing the same analyses with multiple data generating processes, I’m able to make distinctions between artifacts of particular specifications, and more general patterns. Fourth, I introduce a number of original analytical results that help to give intuition for my numerical findings.

Furthermore, this paper provides an analysis of strategic nomination (elsewhere, this is often called ‘strategic candidacy’) that runs parallel to its analysis of strategic voting, thus strengthening the bridge between the two literatures. Many of the existing papers on strategic nomination focus on searching for equilibria in the nomination game, and then examining the properties of these equilibria, including the number of candidates.[7] Here, I explore an alternative approach, which

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[3] For example, Chamberlin (1985), Lepelley and Mbih (1994), Kim and Roush (1996), Lepelley and Valognes (2003), and Tideman (2006).

[4] For example, Favardin, Lepelley, and Serais (2002), and Favardin and Lepelley (2006). Pritchard and Wilson (2007) determine whether manipulation is possible, and what the minimum coalition size is for strategic incursion in each case.

[5] Chamberlain (1985) considers only the four candidate case, while Lepelley and Mbih (1994), Favardin et al (2002), Favardin and Lepelley (2006), Lepelley and Valognes (2003), and Pritchard and Wilson (2007) consider only the three candidate case.

[6] Nitzan (1985), Kim and Roush (1996), Smith (1999), Saari (1990) and Kelly (1993) all use an ‘impartial culture’ (IC) model, while Lepelley and Mbih (1994), Favardin et al (2002), and Favardin and Lepelley (2006) use an ‘impartial anonymous culture’ (IAC) model. Tideman (2006) uses a data set consisting of 87 elections. On the other hand, Chamberlin (1985) uses both spatial model and an impartial culture model, Pritchard and Wilson use both IC and IAC, and Lepelley and Valognes (2003) use a somewhat more general model that has IC and IAC as two of its special cases.

[7] Osbourne and Slivinski (1996) focus on nomination equilibria given the plurality and runoff systems, with costs of entry and benefits of winning. Besley and Coate (1997) focus on nomination and voting equilibria given the plurality system. Moreno and Puy (2005, 2009) focus on positional voting rules, and show that plurality is unique among them in

doesn't look for nomination equilibria per se, but rather determines whether there is a upward or downward pressure on the number of candidates in any given situation,[8] aside from candidates entering the race for the purpose of winning themselves. This allows us to give numerical answers to the question 'how often is there an incentive for strategic nomination?', which is analogous to the question 'how often is there an incentive for strategic voting?'

I focus on eight relatively well-known single-winner voting rules that I consider to be broadly representative of single-winner rules in general: these are plurality, runoff, Hare, minimax, Borda, Coombs, range voting, and approval voting.

The remainder of this paper is organized as follows: In section two, I define the voting rules and the types of strategic behavior that are at issue, and briefly discuss the strategic incentives created by the plurality system, relative to those created by other single-winner systems. In section three, I describe the models and data that I use to generate elections. In sections four and five, I describe how the voting and nomination strategy simulations are constructed, and in sections six and seven, I discuss their results. In section eight, I present analytical proofs that complement the simulation results. In section nine, I conclude. Tables and graphs are contained in appendices.

2. Preliminary definitions

Notation: Let C be the number of candidates, and V be the number of voters. (I assume that the set of candidates and the set of voters do not overlap.) Let c, x, and y serve as candidate indexes, and let v serve as a voter index. Let w denote the winning candidate. Let R_{vc} be the ranking that voter v gives to candidate c (such that lower-numbered rankings are better), and let U_{vc} be the utility that voter v gets if candidate c is elected. Let $x \succ y$ indicate that x is ranked ahead of y, or preferred to y, depending on context; likewise, let $x \sim y$ indicate that x is given the same ranking as y, or that a voter is indifferent between x and y. Let τ be a tiebreaking vector that gives a unique fractional score $\tau_c \in (0,1)$ to each candidate,[9] and let E be a vector of candidate eliminations, such that E_c is initially set to zero for each candidate c. I assume that if a voter gives equal rankings to two or more

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that it will elect the Condorcet winner among declared candidates when there are three or fewer potential candidates, although it will not necessarily do so when there are four or more. Another approach to strategic nomination is given by Tideman (1987), who shows that the addition or removal of 'clone' candidates may change the result given some voting rules, but not others.

[8] In a result that is somewhat analogous to the Gibbard-Satterthwaite theorem, Dutta et al (2001) show that there will in some cases be downward pressure on the number of candidates (in the present paper, I will call this an incentive for 'strategic exit'), given any 'unanimous' and non-dictatorial ranked ballot voting rule. (Note that this doesn't apply to non-ranked systems like approval or range voting.)

[9] For example, these may be assigned randomly, according to lexicographic or anti-lexicographic order, etc., but in any case they are exogenous, and known by the voters and candidates before the election.

candidates, his vote is cast as the average of all strict rankings that can be formed by resolving expressed indifferences; for example, an $\mathrm{A} \succ \mathrm{B} \sim \mathrm{C}$ vote is treated as one half of a $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$ vote, and one half of a $\mathrm{A} \succ \mathrm{C} \succ \mathrm{B}$ vote.

2.1. Voting rule definitions

2.1.1. Plurality: Each voter votes for one candidate, and the candidate with the most votes wins. To facilitate comparison with other methods, I will analyze plurality as a ranked ballot system that awards one point to the candidate listed at the top of each voter's rankings, and zero to the rest.

The formal (ranked ballot) definition of plurality is as follows:

$$f_{vc} = 1\{R_{vc} = 1\}, \forall v, c. \qquad F_c = \textstyle\sum_{v=1}^{V} f_{vc} + \tau_c, \forall c. \qquad w = \mathrm{argmax}(F).$$

Here, f is a V by C matrix that keeps track of individual voters' first choice votes, and F is a length-C vector of the candidates' totals of first choice votes.

2.1.2. Two-round runoff: Each voter chooses one candidate, and the two candidates who receive the most votes compete in a runoff election.

2.1.3. Hare:[10] (Also known as the alternative vote, or instant runoff voting.) Each voter ranks the candidates in order of preference. The candidate with the fewest first choice votes (ballots ranking him ahead of all other candidates in the race) is eliminated. The process repeats until one candidate remains.

Formally, in each round $r = 1, \dots, C-1$, Hare performs the following calculations:

$$f_{vc} = 1\{[E_c = 0] \wedge [R_{vc} < R_{vx}, \forall x\colon (E_x = 0 \wedge x \neq c)]\}, \forall v, c. \qquad F_c = \textstyle\sum_{v=1}^{V} f_{vc} + \tau_c + E_c, \forall c.$$

$$z = \mathrm{argmin}(F). \qquad E_z = \infty. \qquad \text{After round } C-1,\ w = \mathrm{argmin}(E).$$

2.1.4. Coombs:[11] This method is the same as Hare, except that instead of eliminating the candidate with the fewest first-choice votes in each round, it eliminates the candidate with the most last-choice votes in each round.

2.1.5. Minimax:[12] Before defining minimax, it is helpful to define a few related concepts. A **pairwise comparison** is an imaginary head-to-head contest between two candidates, in which each

[10] This system is the application to the single-winner case of proportional representation by the single transferable vote, which is often named for Thomas Hare, who introduced the idea of successively eliminating the candidate with the fewest first choice votes. See Hoag and Hallett (1926, 162-95).

[11] See Coombs (1964). I note that some version of Coombs include a provision to automatically elect a candidate who holds a majority of first choice votes, but I don't use this provision here.

[12] Black (1958), page 175, develops the minimax method as a possible interpretation of Condorcet's intended proposal. Levin and Nalebuff (1995) label this method as the "Simpson-Kramer min-max rule"; presumably this is due to Simpson (1969) and Kramer (1977). Nurmi (1999) refers to it as "Condorcet's successive reversal procedure", on page 18. Tideman (2006) refers to it as "maximin", on page 212.

voter is assumed to vote for the candidate whom he gives a better ranking to. A **Condorcet winner** is a candidate who wins all of his pairwise comparisons. A **majority rule cycle** is a situation in which each of the candidates suffers at least one pairwise defeat, so that there is no Condorcet winner.[13] A **Condorcet method** (or a Condorcet-efficient voting rule), is any single-winner voting rule that always elects the Condorcet winner when one exists. Minimax is a Condorcet method that uses ranked ballots, and works as follows: each candidate receives a score equal to the greatest number of voters who oppose him in any pairwise comparison, and the candidate who receives the lowest score is the winner.

Condorcet winners and majority rule cycles can be formally defined in this way: Let P be the **pairwise matrix**, such that P_{xy} gives the number of voters who rank candidate x ahead of candidate y. Let p be the 'pairwise array', which is used to calculate the pairwise matrix as follows:

$$p_{xyv} = 1\{R_{vx} < R_{vy}\}, \forall x, y, v. \qquad P_{xy} = \sum_{v=1}^{V} p_{xyv}, \forall x, y.$$

A Condorcet winner is a candidate x such that $P_{xy} > P_{yx}, \forall y$. A majority rule cycle is a situation in which $\forall x, \exists y\colon P_{yx} > P_{xy}$.

The minimax winner can be found as follows: $\mathrm{M}_y = \max_{x=1}^{C} P_{xy} - \tau_y$. $\qquad w = \operatorname{argmin}(\mathrm{M})$. That is, in the case of a cycle, minimax elects the candidate with the weakest pairwise defeat.

2.1.6. Borda count:[14] Each voter ranks the candidates in order of preference. Each first-choice vote is counted as C points; each second-choice vote as C-1 points, and so on. The winner is the candidate with the most points. Equivalently, each candidate may receive one point for each candidate who is ranked above him on each ballot; the winner in this case is the candidate with the fewest points.

Using the latter definition (which is more convenient for programming purposes), we can find the Borda winner using the pairwise matrix as follows: $\mathrm{B}_y = \sum_{x=1}^{C} P_{xy} - \tau_y$. $\qquad w = \operatorname{argmin}(\mathrm{B})$.

2.1.7. Approval voting:[15] Each voter chooses whether or not to 'approve' each candidate; that is, to give each candidate one point or zero points. The winner is the candidate with the most points.

2.1.8. Range voting: Each voter may give each candidate any real number of points within a specified range (e.g. 0 to 1 or 0 to 100). The winner is the candidate with the most points.

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[13] Condorcet (1785) describes the pairwise comparison method and the Condorcet winner. He also observes the possibility of a majority rule cycle emerging despite transitive voter preferences – this is known as the Condorcet paradox.

[14] This method was proposed by Jean-Charles de Borda in 1770; see McLean and Iain (1995), page 81. Saari (2001) gives a contemporary argument in favor of it.

[15] See Brams and Fishburn (1978) and Brams and Fishburn (1983).

2.2. Strategy definitions

In the case of ranking-based methods, **strategic voting** means providing a ranking of the candidates that differs from one's true preference ordering. In the case of approval voting or range voting, it means departing from one's sincere cardinal ratings of the candidates.

Two subsidiary types of strategic voting that will provide important analytical distinctions are the 'compromising' and 'burying' strategies.[16] The **compromising strategy** entails voters improving the ranking or rating of a candidate, in order to cause that candidate to win. For example, a voter with sincere preferences $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C} \succ \mathrm{D}$ could compromise in favor of B by voting $\mathrm{B} \succ \mathrm{A} \succ \mathrm{C} \succ \mathrm{D}$. The **burying strategy** entails voters worsening the ranking or rating of one candidate, in order to prevent that candidate from winning, and thus to cause a more preferred candidate to win. For example, a voter with sincere preferences $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C} \succ \mathrm{D}$ could bury C (in order to help A or B) by voting $\mathrm{A} \succ \mathrm{B} \succ \mathrm{D} \succ \mathrm{C}$.

Equivalently, if w is the sincere winner, and q is an alternative candidate whom strategic voters are seeking to elect instead (as they prefer $q \succ w$), the compromising strategy is to give q a better ranking (or rating), and the burying strategy is to give w a worse ranking (or rating).

When citizens cast their votes in a plurality election for candidates they consider to be the 'lesser of two evils', rather than for their sincere favorites, this is an example of the compromising strategy. For example, suppose that 49% of voters have the preference ordering $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$, 24% of voters prefer $\mathrm{B} \succ \mathrm{A} \succ \mathrm{C}$, 24% of voters prefer $\mathrm{B} \succ \mathrm{C} \succ \mathrm{A}$, and 3% of voters prefer $\mathrm{C} \succ \mathrm{B} \succ \mathrm{A}$. (This may be more intuitive if one imagines, for example, that candidate A is a right-of-center candidate, candidate B is a left-of-center candidate, and candidate C is a left-wing candidate.) If all voters vote for their sincere favorites, A wins with 49% of the vote, but if the $\mathrm{C} \succ \mathrm{B} \succ \mathrm{A}$ voters compromise by voting for candidate B, B wins with 51% of the vote.

To see an example of the burying strategy, suppose that voters have the same preferences as above, but that the election method is Borda or minimax instead of plurality. The sincere winner is candidate B, but if the $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$ voters all bury B by voting $\mathrm{A} \succ \mathrm{C} \succ \mathrm{B}$, the winner is A.

Strategic nomination means non-winning candidates entering or leaving a race in order to change the outcome to one they prefer; I describe these as **strategic entry** and **strategic exit**, respectively. The practice of strategic nomination can be seen in the party primaries that are a regular feature of American politics. That is, if two or more candidates with similar views run in the same plurality election, the voters who support those views will be divided among them, giving an

[16] This terminology was used by Blake Cretney, on his currently-defunct web site Condorcet.org.

advantage to other candidates with opposed views. Therefore, it is practical for groups of fairly like-minded people to form some kind of association (a political party) which fields only one candidate per election (asking the rest to strategically exit), and which provides some kind of process (a primary) for deciding who this one candidate should be.

2.3 Preview of results, and their significance: In this paper, I find that plurality has more frequent incentives for the compromising strategy, and for strategic exit, than any of the other voting rules examined here. Since strategic exit gives third party candidates a disincentive to run, and frequent use of the compromising strategy gives voters a disincentive to support third party candidates who do run, these phenomena together may provide much of the explanation for Duverger's Law,[17] which states that countries using the plurality voting rule will tend to have two dominant political parties.[18] Therefore, switching to one of the alternative systems described here could decrease the extent to which a two party system prevails, and thus increase the competitiveness of elections.

However, I do not find that plurality is most vulnerable to strategic voting overall; instead, I find that range voting, Coombs, approval voting, and Borda are significantly more likely to be vulnerable given most specifications. Although these methods create less frequent incentives for compromising, they create frequent incentives for burying, whereas plurality is immune to burying (as are two round runoff and Hare).[19] Also, I find that Borda, not plurality, gives the most frequent incentives for strategic entry. Whereas the effects of compromising and strategic exit are relatively well-understood by virtue of the long history of the plurality system, adopting a voting rule that creates frequent burying or strategic entry incentives would bring us into somewhat unknown territory.

3. Models and data

3.1. Spatial voting model:[20] The spatial voting model used here distributes both voters and candidates randomly in S-dimensional issue space, according to a multivariate normal distribution without covariance. Voters are assumed to prefer candidates who are closer to them in this space.

[17] See Duverger (1964).

[18] To be precise, the strong incentives for compromising and strategic exit make it highly likely that there will only be two 'viable' candidates in any particular plurality election. The two parties that these candidates belong to may vary from jurisdiction to jurisdiction, but once two parties have been established as dominant in any given jurisdiction, it is difficult for a third party to mount an effective challenge there.

[19] I discuss these immunities in section 8 below. Woodall (1997) demonstrates that Condorcet-efficient methods can't satisfy his 'later-no-help' and 'later-no-harm' criteria; a similar proof can be used to show that they must be vulnerable to the burying strategy in some situations.

[20] This is similar to the model described in Chamberlin and Cohen (1978), but without inter-dimensional covariance.

Formally, we create L and Λ matrices that give the voter and candidate locations, respectively:

$$L_{vs} \sim \mathcal{N}(0,1), \forall v, s. \qquad \Lambda_{cs} \sim \mathcal{N}(0,1), \forall c, s. \qquad U_{vc} = -\sqrt{\sum_{s=1}^{S} (L_{vs} - \Lambda_{cs})^2}, \forall v, c.$$

3.2. Impartial culture model:[21] The impartial culture model used here simply treats each voter's utility over each candidate as an independent draw from a uniform distribution, thus making each ranking equally probable, independent of other voters' rankings. Formally, $U_{vc} \sim \mathcal{U}(0,1), \forall v, c$.

3.3. ANES Time Series Study: I use the June 24, 2010 version of the Time Series Cumulative Data File, published by the American National Election Studies project. In particular, I use the 'political figure thermometer' questions (asking respondents to rate particular politicians on a scale from 0 to 100) from 1968 to 2008, to which there are 21,474 responses. The list of politicians varies from year to year; current presidents and vice-presidents are always included, as are Democratic and Republican presidential and vice-presidential candidates. In addition, there are various other figures who are included in the survey even when they don't hold any of these positions, e.g. Ted Kennedy (from 1970 to 1988), Ronald Reagan (from 1968 to 1990, and again in 2004), Hillary Clinton, Ross Perot, etc. For the purposes of these simulations, I treat the thermometer ratings as the voters' sincere cardinal ratings of the candidates, and use them to derive sincere ordinal preferences.[22]

In some presidential election years, respondents are rating as many as 12 politicians; in others, as few as 7. For a given number of candidates C, I generate $\binom{A_t}{C}$ imaginary contests (i.e. all of the possible C-candidate subsets) in each presidential election year, where A_t is the number of politicians rated by survey respondents in year t.[23] In each of these simulated elections, I treat each survey respondent who gives a strict ranking of the C candidates as one voter; although the data set includes weighting variables in some years, I don't make use of them. To get the score for each year t, I determine how many of these $\binom{A_t}{C}$ elections are vulnerable to strategic manipulation. I then take the average over these yearly scores to get the overall score for the given value of C.

4. Strategic voting simulation design

4.1. How often is sincere voting a core equilibrium? (analysis V1)

My primary approach to strategic voting is as follows: Given a particular voting rule and preference profile, I begin with sincere votes, and ask whether there is a group of voters who can

[21] This is defined, for example, in Nitzan (1985).
[22] Here, I follow Tideman and Plassmann (2011).
[23] This process creates 931 elections with three candidates, 1553 with four candidates, 1898 with five candidates, and 1764 with six candidates.

change the result from the sincere winner to one whom they all prefer, by changing their votes. If so, the election is manipulable. If not, sincere voting is a core equilibrium.

As it turns out, it is difficult to test for core equilibria in strategic voting using brute force. For example, given a ranking-based method, there are $C!$ possible rankings of C candidates, and thus $C!^V$ ways in which V voters can rank them. Thus, it can be a daunting task to search over every one of these ranking profiles to determine whether any of them give an advantage to all of the voters whose votes differ from their sincere preferences. Therefore, I've written separate algorithms to determine whether each of the eight voting rules is vulnerable to manipulation. To give a sense of how these operate, I describe then briefly below.[24]

In these descriptions, let q be a potential winner by strategy, let $\psi_v = 1\{U_{vq} > U_{vw}\}$ indicate whether voter v prefers candidate q to the sincere winner w, and let $\Psi = \sum_{v=1}^{V} \psi_v$ be the number of potential strategists. Also, let a tilde mark indicate a version of an existing variable that is altered by omitting these potential strategists; for example, $\widetilde{P_{xy}} = \sum_{v=1}^{V}\left(1\{\psi_v = 0\} \cdot p_{xyv}\right), \forall x, y$. Let $\Omega_q = 1$ indicate that manipulation on behalf of candidate q is feasible, and let $\Omega_q = 0$ indicate that it is not.

4.1.1. Plurality: First, I calculate the sincere winner w using the first choice votes vector F, and I find the pairwise matrix P, as described in subsections 2.1.1 and 2.1.5. Then, I loop through possible strategic challengers $q \neq w$ to determine whether q would win if all the voters who prefer q to w voted for q; this is the necessary and sufficient condition for strategic incursion on behalf of q to be possible. Formally, $\Omega_q = 1 \leftrightarrow P_{qw} + \tau_q > F_w$.

4.1.2. Approval voting: In my simulations, I suppose that voters' sincere inclination is to approve candidates who give them greater than average utility. (Alternative assumptions are possible, but this seems as straightforward as any.) Strategic incursion is possible on behalf of a challenger candidate $q \neq w$ if and only if q wins when all of the voters who prefer q to w vote to approve q and no one else.

4.1.3. Range voting: To find the sincere winner, I convert voter utilities into sincere ratings, normalized so as to span the [0,1] interval, and then sum these ratings, using the tiebreaker if necessary. As with approval, we can determine whether manipulation on behalf of q is possible by checking whether q wins when all those who prefer q to w give the maximum rating to q, and the minimum rating to all other candidates.

[24] More detailed descriptions, along with the codes themselves, are available from the author on request.

4.1.4. Two round runoff: To cause $q \neq w$ to win in the runoff system, strategists must cause the runoff to be between q and some other candidate d, whom q can beat. Therefore, within the loop over $q \neq w$, the program loops over $d \neq q$, and determines whether (1) those who prefer q to d or q to w (or both) constitute a majority, enabling q to win the runoff, and (2) the strategists can cause q and d to be the top two finishers in the first round. Strategic incursion is possible if and only if both of these are true.

4.1.5. Hare: The Hare program is somewhat similar to the two round runoff program, but more complex. To determine whether those who prefer a given candidate q can change their votes so that q is elected, I examine each of the $(C-1)!$ elimination orders that result in q's victory, and determine whether the $q \succ w$ voters can cause any of them to occur. To determine whether an elimination order is feasible, I examine each of the rounds from $r = 1, \ldots, C-1$, continuing as long as the strategists can cause the elimination of the candidate who is supposed to be eliminated in round r. In determining this, I need to keep track of votes that strategists must commit to particular candidates in order to ensure a given elimination, and bind them to these votes until those candidates are also eliminated.

4.1.6. Coombs: The structure of this program is similar to that of the Hare program, although it is a bit less complex, as the strategists' best chance to achieve a given elimination order is always to simply focus all of their last choice votes on the candidate they want to eliminate in each round.

4.1.7. Minimax: To determine whether minimax is vulnerable to strategic manipulation on behalf of some candidate q, I begin by finding the nonstrategic pairwise matrix $\tilde{P}$, the corresponding minimax scores $\widetilde{\mathrm{M}}$, and the value T, which is defined as q's entry in $\widetilde{\mathrm{M}}$. Formally,

$$\widetilde{p_{xyv}} = 1\{\psi_v = 0\} \cdot 1\{U_{vx} > U_{vy}\}, \forall x, y, v. \qquad \widetilde{P_{xy}} = \textstyle\sum_{v=1}^{V} \widetilde{p_{xyv}}, \forall x, y.$$

$$\widetilde{\mathrm{M}_y} = \max_{x=1}^{C} \widetilde{P_{xy}} - \tau_y, \forall y. \qquad \mathrm{T} = \widetilde{\mathrm{M}_q}.$$

Because strategic voters can do nothing to reduce T, they must arrange for all of the other candidates to have higher (worse) scores in order to elect q. This means that each of the other candidates needs to have a certain number of votes against him in at least one pairwise contest. I proceed by giving separate consideration to each of several possible 'defeat profiles', which, for each candidate $y \neq q$, names another candidate who will give y a pairwise beat stronger than T.

Given a strategist candidate q, and given a particular defeat profile, I create a 'need' matrix η, such that η_{xy} tells us how many votes the strategists need to add to x's side of the x vs. y pairwise contest, once the non-strategists' votes have already been taken into account. If the defeat profile does not create a cycle, the number of strategists needed is simply the largest value in η. If it does create a cycle of K defeats, whose entries in η sum to Σ, the number of strategists needed is given by the greatest of these entries, or by $\frac{\Sigma}{\mathrm{K}-1}$, whichever is larger. (For example, $\eta_{1,2} = 5$, $\eta_{2,3} = 6$, and $\eta_{3,1} = 7$ forms a loop with three beats, and the number of strategists needed to ensure the defeat of all three candidates in the loop is $\frac{5+6+7}{3-1} = 9$.)

4.1.8. Borda: $\tilde{p}$ and $\tilde{P}$ are calculated as above. Then, $\widetilde{\mathrm{B}_y} = \sum_{x=1}^{C} \widetilde{P_{xy}} - \tau_y, \forall y$ gives the Borda scores from non-strategic voters, and $\Phi = \widetilde{B_q}$ gives the minimum Borda score of q, which strategists can't reduce. The strategists' goal is to form their own 'strategic pairwise matrix' $\hat{P}$, such that q is the winner according to the combined pairwise matrix $P' = \hat{P} + \tilde{P}$, which requires that $\sum_{x=1}^{C} \widehat{P_{xy}} + \sum_{x=1}^{C} \widetilde{P_{xy}} - \tau_y > \Phi, \forall y \neq q$.

In short, the method of searching for a successful $\hat{P}$ is as follows. $\hat{P}$ begins as a matrix of zeros, and then is updated so that $\widehat{P_{qy}} = \Psi, \forall q \neq y$. If there are any 'covered' candidates $c \neq q$ such that $\sum_{x=1}^{C} P'_{xc} - \tau_c > \Phi$, the strategists 'lift' them, i.e. rank them between q and the remaining candidates. If this causes other candidates to be covered in turn, then they are lifted as well, though they are still ranked behind candidates who were lifted earlier.

If the iteration of this process leads to every candidate being covered, then $\Omega_q = 1$. Otherwise, strategic voters are committed, one at a time, to ranking the remaining uncovered candidates as tied for last choice. If and when this process causes additional candidates to be covered, they are lifted as well, by the strategists who haven't yet committed to ranking them as tied for last. This process continues until all candidates other than q are covered, in which case $\Omega_q = 1$, or until the supply of strategists is exhausted, in which case $\Omega_q = 0$.

4.2. How often can simple strategies succeed? (analysis V2)

4.2.1. Compromising and burying together: We have seen that some strategies are highly complex, and require both precise knowledge of other voters' preferences and precise coordination to be successful. Thus, as a complement to the primary analysis, it might be interesting to know how often each method is vulnerable to simpler voting strategies.

This analysis works as follows: I begin by finding the sincere winner, w. Then, for all other candidates $q \neq w$, I check to see whether q would win if the $q \succ w$ voters were to change their ballots so as to give the best possible ranking or rating to q, and the worst possible ranking or rating to w. Certainly, there may be other 'simple' strategies, but this is one of the more obvious ones, it has the advantage of being applicable to all of the voting methods we're examining, and as we'll see below, it can succeed in most of the cases in which strategy is possible.

4.2.2. Compromising: For all non-winning candidates $q \neq w$, I check to see whether q would win if the $q \succ w$ voters were to change their votes so as to give q the best possible ranking or rating.

4.2.3. Burying: For all non-winning candidates $q \neq w$, I check to see whether q would win if the $q \succ w$ voters were to change their votes so as to give w the worst possible ranking or rating.

5. Strategic nomination simulation design

In order to provide a relative measure of how frequently different voting methods will have incentives for strategic nomination, I start with the assumption that there are C_I candidates who are in the race by default, and C_O candidates who are out of the race by default, but who would be prepared to enter it. (Thus, there are $C = C_I + C_O$ potential candidates overall.)

The V by C matrix of voter utilities over candidates is generated as before, using the spatial model. In addition to this, I generate a C by C matrix of candidates' preferences Υ, such that $\Upsilon_{xy} = -\sqrt{\sum_{s=1}^{S}\left(\Lambda_{xs} - \Lambda_{ys}\right)^2}$ gives the utility that candidate x experiences if candidate y wins. This definition of Υ implies that all candidates prefer their own election to the election of any other candidate. I focus on the spatial model because, of the three data generating processes I use, it gives us the most natural means of calculating candidates' preferences over other candidates.

Starting from the default sets of 'in' candidates and 'out' candidates, I ask whether any individual candidate[25] can change the result to his benefit by either leaving the race (strategic exit), or entering it, but not winning (strategic entry). In this analysis, I assume that voting is sincere.

I omit approval and range from these simulations because they would seem to have at most a minimal susceptibility to strategic nomination, as I define it here. For example, if sincere ratings are defined over the set of all C potential candidates, rather than only the C_I candidates currently in the race, then approval and range are not susceptible to strategic nomination at all.

[25] It is also possible to ask how often groups of candidates may change the result in a way that they all prefer by either simultaneously entering or simultaneously exiting (thus, looking for core equilibria instead of Nash equilibria), but I omit this additional analysis in the interest of brevity, because its results don't differ in any particularly interesting way from the results of the analysis already described.

6. Strategic voting results

6.1. General voting strategy analysis

Tables 2-5 and figures 1-3 give the results of the general voting strategy analysis using the spatial model, the impartial culture model, and the ANES data set. Each data point indicates the share of trials in which a group of voters can change the result to all of its members' mutual benefit by voting insincerely. I use $10{,}000$ trials for each (non-ANES) data point, which causes the margin of error to be $.0098$ or less, with 95% confidence.[26]

Using the spatial model, the ANES data, and the impartial culture model with relatively few voters, there is a clear stratification between Hare and runoff, which are vulnerable to manipulation with low frequency, minimax and plurality, which are vulnerable to manipulation with moderate frequency, and approval, Borda, range, and Coombs, which are vulnerable to manipulation with high frequency. Within these groups, Hare is almost always better than runoff, and minimax is almost always better than plurality.

In the impartial culture model, when the number of voters is large, most of the eight methods are vulnerable approximately 100% of the time, but Hare and runoff are not. The manipulability of runoff is not close to 100% when $C = 3$, but it is close to 100% when $C \geq 4$. The manipulability of Hare is not close to 100% for any $C = 3, 4, 5, 6$. Propositions 4-6 below provide some intuition for these results. Increasing S generally decreases manipulability in most systems, particularly Hare and runoff, but it has the opposite effect on Coombs. Table 5 shows that, in the spatial model, changes in V have only a modest impact.

These results are broadly consistent with the existing literature on coalitional manipulation. For example, out of plurality, Borda, Hare, and Coombs, Chamberlin (1985) finds that Borda is most manipulable, and Hare is least manipulable, in both the impartial culture model and a spatial model. Lepelley and Mbih (1994) use an impartial anonymous culture (IAC) model,[27] and find the following ordering of methods from least to most manipulable: Hare, plurality, Coombs. Also using an IAC model, Favardin and Lepelley (2006) find the ordering Hare, runoff, minimax, plurality, Coombs, Borda (in what they call case 3, which is closest to the analysis here). Lepelley and Valognes (2003) uses a more general model that includes both IC and IAC, and finds the ordering Hare, minimax, plurality, Coombs, Borda, with numerical scores that are close to the ones presented

[26] A margin of error of ±.0098 is the upper bound, which applies when the true probability is exactly one half. I further reduce the random error in the difference between the scores that the various voting methods receive by using the same set of randomly generated elections for each method.

[27] Despite the similar name, this is not equivalent to the impartial culture model. Rather, the IAC model supposes that every possible division of the voters among the $C!$ possible preference orderings is equally likely.

here when the specifications overlap. Tideman (2006) uses a data set consisting of 87 elections, and finds the ordering Hare, minimax, runoff, plurality, Borda, range, approval.

6.2. Simple strategies results

6.2.1. Compromising and burying together: The primary result of analysis V2 is that it gives very similar numbers to analysis V1. This tells us that the simple combination of compromising and burying tends to be quite effective, and that most examples of strategic vulnerability do not require voters to orchestrate very complex manipulation schemes to be successful.

Therefore, to avoid redundancy, table 6 gives only a smattering of the direct results of analysis V2, with an emphasis on specifications with large numbers of candidates (which are computationally expensive for V1). More complete results are indirectly summarized in table 7, which compares the frequency of strategic opportunities using the simple strategy to the overall frequency with which manipulation is logically possible. The last column of the table shows that taking all of the voting rules together, approximately 94% of the strategically vulnerable cases in the spatial model, 94% of the vulnerable cases in the impartial model, and 97% of the vulnerable ANES elections are also vulnerable to this simple strategy.

Hare and runoff are alone in not being vulnerable to the simple strategy in over 90% of cases such that manipulation is possible. Perhaps this is because many of the remaining manipulable cases require a ‘push-over’ strategy[28] that adds early-round first choice support to an otherwise weak candidate who will be relatively easy for the strategic candidate to defeat in the last round.

6.2.2. Compromising strategy results: Tables 8-10 and figures 4-6 show the voting rules’ vulnerability to the compromising strategy, given various specifications. As discussed in subsection 8.2, Coombs is immune to the compromising strategy. As shown in proposition 1, minimax is immune to compromising when there is a sincere Condorcet winner; this helps to explain why it is next-least vulnerable to compromising after Coombs. Plurality is the most vulnerable to compromising in all specifications, and approval, range, and Borda are consistently more vulnerable than Hare and runoff.

6.2.3. Burying strategy results: Tables 11-13 and figures 7-9 show the voting rules’ vulnerability to the burying strategy, given various specifications. As discussed in subsection 8.1, plurality, runoff, and Hare are immune to the burying strategy. Coombs, range voting, and approval voting

[28] This was also defined by Blake Cretney on Condorcet.org. Note that it is not the same as burying, compromising, or a combination of these, as it involves shifting votes from candidate q to candidate x in order to help candidate q.

are consistently the most vulnerable to burying, while Borda and minimax form an intermediate category.

7. Strategic nomination results

Tables 14-15 and figures 10-11 show the voting rules' vulnerability to strategic exit and entry. I find that plurality is most frequently vulnerable to strategic exit (proposition 9 gives some intuition for this result), although with large numbers of candidates in the race, runoff and Hare are vulnerable with similar frequency. I find that Borda is most vulnerable to strategic entry, and that Coombs is second-most vulnerable. (Propositions 10 and 11 give some intuition for these results.) Minimax is vulnerable to both exit and entry with very low frequency, because its vulnerability depends on the existence of a majority rule cycle, as demonstrated in propositions 7 and 8.

8. Analytical results

Notation: Let an overscore denote that a variable is defined with respect to voters' sincere preferences; for example, the sincere ranking matrix is $\bar{R}$, the sincere pairwise matrix is $\bar{P}$, etc.

8.1. Burying strategy

Discussion: It is fairly easy to see that plurality, runoff, and Hare are immune to the burying strategy. That is, in each of these systems, if voters who prefer $q \succ w$ rank w still lower, it cannot affect w's chances of winning unless q has already been eliminated. Meanwhile, minimax, Borda, approval, range, runoff, and Coombs are vulnerable to the burying strategy.

8.2. Compromising strategy

Discussion: It is also fairly easy to see that Coombs is immune to the compromising strategy, using similar logic. Incidentally, the 'anti-plurality' system, which elects the candidate with the fewest last choice votes, is another method that has this property. Plurality, runoff, Hare, minimax, Borda, approval, and range are all vulnerable to compromising.

Proposition 1: If there is a sincere Condorcet winner, a Condorcet method is not vulnerable to the compromising strategy.

Proof: If voter v prefers q to w, then q is ahead of w in v's sincere rankings, which means that the q, v, w entry in the sincere pairwise array is 1. Formally, $U_{vq} > U_{vw} \leftrightarrow \overline{R_{vq}} < \overline{R_{vw}} \leftrightarrow \overline{p_{qvw}} = 1$. If voter v gives q a still-better ranking, this will not change P_{qw}, because v's q, w ordering will be unchanged; nor will it change any other P_{cw}, because w isn't moving in q's ranking relative to any other candidate. Formally, $p'_{cwv} = 1 \leftrightarrow \overline{p_{cwv}} = 1, \forall c, v$, which implies that $P'_{cw} = \overline{P_{cw}}, \forall c$.

Because w is the sincere Condorcet winner, $\overline{P_{wc}} > \overline{P_{cw}}, \forall c \neq w$. Combining this with the above, $P'_{wc} > P'_{cw}, \forall c \neq w$, which implies that w is still the Condorcet winner and therefore the election winner. ■

Proposition 2: Plurality, runoff, Hare, and minimax are vulnerable to the compromising strategy when there is a sincere majority rule cycle.

Proof: If there is a sincere majority rule cycle, then for any given sincere winner w, there will be some alternative candidate q such that a majority prefers q to the sincere winner w. Formally, $\exists q: \overline{P_{qw}} > \overline{P_{wq}}$. If all $q \succ w$ voters rank q as their first choice, then q will be the winner in plurality, runoff, Hare, and minimax. ■

Discussion: From propositions 1 and 2, we see that (in the absence of pairwise ties, which are unlikely in large elections[29]) plurality, runoff, and Hare are vulnerable to compromising whenever a Condorcet method is vulnerable to compromising, but a Condorcet method is not necessarily vulnerable to compromising when plurality, Hare, or runoff is vulnerable to compromising. Furthermore, though proofs are omitted here, it can be shown that if plurality isn't vulnerable to compromising in a particular election, then neither runoff nor Hare will be vulnerable to compromising in the same election. Thus, these two methods dominate plurality with respect to compromising incentives, while all three are almost entirely dominated by any Condorcet method.

8.3. General voting strategy

Proposition 3: If the sincere Condorcet winner w is also the sincere first choice of more than one third of the voters, then both Hare and runoff will elect w and be non-manipulable.

Proof: a candidate with more than $\frac{V}{3}$ first choice votes will not be eliminated before the last round, because each of the prior rounds will include three or more candidates, and because none of the voters whose sincere first choice is w will have an incentive to defect on behalf of an alternative candidate q. Because the last round is equivalent to a pairwise comparison, and because w is the Condorcet winner, w will win. ■

Note: This property is not shared by plurality, minimax, Borda, Coombs, approval, or range.

'Almost-symmetrical preferences' definition: For the sake of propositions 4-6, define 'almost-

[29] See for example Beck (1975), who shows that ties are unlikely in large two-candidate elections; it is not hard to extend this intuition to the case of a pairwise tie within a multi-candidate election.

symmetrical preferences' as follows: Let the candidates be $x_1, \ldots, x_C$, let there be N voters with each possible preference ordering, plus one voter with an $x_1 \succ \cdots \succ x_C$ preference ordering.

Discussion: The purpose of the following propositions 4 through 6 is to shed some light on a dynamic observed in the impartial culture simulation results: Given $C \geq 4$, Hare is the only one of these eight voting methods that doesn't converge towards 100% manipulability as the number of voters gets large. Given $C = 3$, this property is shared by the runoff system. When V is large in an impartial culture model, each preference order appears in approximately equal proportion; therefore, the winner's margin of victory tends to be very small relative to the number of voters who prefer an alternative candidate to the sincere winner. These features are captured in the 'almost-symmetrical preferences' scenario that provides the basis for these propositions.

Proposition 4: Given the Hare system, and a set of almost-symmetrical preferences, x_1 will be the winner, and the result will not be manipulable.

Proof: In the first round of counting, $\overline{F_1} = N(C-1)! + 1$, and $\overline{F_c} = N(C-1)!$, $\forall c \neq 1$. That is, the votes are divided evenly, except for the one extra vote that gives x_1 his advantage. Strategists can't cause x_1 to be eliminated in the first round, because x_1 is the first choice of more than $\frac{V}{C}$ voters, which means that it's not possible for strategists to arrange for all of the other candidates to have more votes.

Similar logic holds in later rounds of counting. That is, when D candidates remain, no candidate can have the fewest first choice votes if he has at least $\frac{V}{D}$ votes; because x_1 will have $\frac{V-1}{D} + 1 = \frac{V+D-1}{D}$ sincere votes, he can't be eliminated. Because x_1 is their first choice among non-eliminated candidates, none of these voters will be interested in participating in a strategy on behalf of any non-eliminated candidate q. Therefore, strategic incursion against x_1 can't succeed. ■

Proposition 5: Given almost-symmetrical preferences, the result will be manipulable in plurality, minimax, Borda, and Coombs, given a sufficiently large value of N.

Plurality: With $\frac{V-1}{C} + 1$ first choice votes (where $V = NC! + 1$), to every other candidate's $\frac{V-1}{C}$ first choice votes, x_1 is the sincere winner. For any $q \neq 1$, the number of potential $x_q \succ x_1$ strategists is given by $\overline{P_{q1}} = \frac{V-1}{2}$, which is greater than $\overline{F_1} = \frac{V+C-1}{C}$, for $V > \frac{3C-2}{C-2}$. Therefore, if these strategists vote for x_q, x_q will win.

Minimax: For $i < j$, $\overline{P_{ij}} = \frac{1}{2}NC! + 1$, and for $i > j$, $\overline{P_{ij}} = \frac{1}{2}NC!$. Therefore, for $j = 1$, $\max_{i=1}^{C}\overline{P_{ij}} =$

$\frac{1}{2}NC!$, and for $j \neq 1$, $\max_{i=1}^{C}\overline{P_{ij}} = \frac{1}{2}NC! + 1$. So, x_1 is the sincere winner. Suppose, however, that all of the voters who prefer $x_2 \succ x_1$ vote $x_2 \succ x_3 \succ \cdots \succ x_C \succ x_1$. Then, we still have $P'_{1,2} = \frac{1}{2}NC! + 1$, but now, $P'_{c1} = \frac{2}{3}NC!, \forall c \geq 3$, and $P'_{cd} = \frac{2}{3}NC! + 1, \forall c, d{:}\, c < d \wedge c \geq 2$. Therefore, there will be a majority rule cycle, but the defeat against x_2 will be the weakest, so x_2 will win.

Coombs: Given sincere voting, the elimination order will be $x_N, x_{N-1}, \dots, x_2$, so that x_1 is the winner. However, if the voters who prefer $x_2 \succ x_1$ change their votes so that x_1 is ranked last, but the rankings are otherwise the same, x_1 will be eliminated in the first round (with $\frac{1}{2}NC!$ last choice votes, which is greater than x_N's total of $\frac{1}{C}NC! + 1$ last choice votes), and then the remaining eliminations will occur in their original order, leaving x_2 as the winner.

Borda: The sincere Borda score for each candidate j is $\overline{\mathrm{B}_j} = \frac{1}{2}NC!\,(C-1) + j - 1$, and the sincere winner is x_1. However, if all of the voters who prefer $x_2 \succ x_1$ vote $x_2 \succ x_3 \succ \cdots \succ x_C \succ x_1$, the resulting Borda scores will be $\mathrm{B}'_1 = NC!\left[\frac{1}{2} + \frac{2}{3}(C-2)\right]$, $\mathrm{B}'_2 = NC!\left[\frac{1}{2} + \frac{1}{3}(C-2)\right] + 1$, and $\mathrm{B}'_j = NC!\left[\frac{1}{3} + \frac{2}{3}(j-2) + \frac{1}{3}(C-j)\right] + j - 1, \forall j \neq 1, 2$. Therefore, the new winner will be x_2. ■

Proposition 6: Given the runoff system and almost-symmetrical preferences, the result will be non-manipulable if $C \leq 3$, but manipulable if $C \geq 4$.

Case 1, $C = 3$: Because x_1 is the sincere Condorcet winner, and because x_1 has more than $\frac{V}{3}$ votes, x_1 is the sincere winner, and the result is non-manipulable, by proposition 3 above.

Case 2, $C \geq 4$: As in the $C = 3$ case, x_1 is the sincere winner. x_1 has $\frac{1}{C}NC! + 1$ first choice votes, and there are $\frac{1}{2}NC!$ voters who prefer $x_2 \succ x_1$. Aside from these, x_3 has $\frac{1}{2C}NC!$ first choice votes. The $x_2 \succ x_1$ voters can prevent x_1 from reaching the runoff by distributing their votes between x_2 and x_3 so that x_1 has fewer votes than each of them. This requires $\left[\frac{1}{2C}NC! + 1\right] + \left[\frac{1}{C}NC! + 1\right] = \frac{3}{2C}NC! + 2$ votes, which is less than the number of $x_2 \succ x_1$ voters, so the strategy is feasible. ■

Remarks on core equilibrium existence: Proofs are omitted here for brevity, but it is possible to show that the existence of a sincere Condorcet winner is a necessary condition for core equilibrium existence in the voting game given any of these eight methods. In the cases of plurality, runoff, Hare, minimax, Coombs, approval, and range, it is also a sufficient condition – the core may well not include universally sincere voting, but it will at least include a rather artificial voting profile in which everyone gives maximum scores to the sincere Condorcet winner (and minimum scores to other candidates in the case of approval and range). Interestingly, the existence of a Condorcet winner is not a sufficient condition for core equilibrium existence in Borda, though a stronger version of this is a sufficient condition, i.e. a case in which there is a candidate who has more than $\frac{2C-2}{3C-2}V$ voters on his side in each pairwise comparison.

8.4. Strategic nomination

Proposition 7: A Condorcet method is not vulnerable to strategic exit if there is a Condorcet winner among the candidates initially on the ballot.

Proof: If w is a Condorcet winner, then w will also be the election winner. If candidate q exits the race, the pairwise contests between the remaining candidates will not be changed. Therefore, w will still be the Condorcet winner, and the election winner. ■

Proposition 8: A Condorcet method is not vulnerable to strategy entry, unless the final ballot (after entry) lacks a Condorcet winner.

Proof: If the final ballot has a Condorcet winner, then this candidate must be the election winner. By definition of strategic entry, none of the newly-entered candidates may be the winner. Therefore, the winning candidate is a candidate w who was on the old ballot, and who has pairwise defeats against every other candidate on the new ballot. Because all candidates on the new ballot are also on the old ballot, w has pairwise defeats against all candidates on the old ballot. Therefore, w is the winner given the old ballot as well as the new ballot. ■

Proposition 9: Given the plurality rule (ranking version), if the electorate consists of A voters whose preferences are $x_1 \sim \dots \sim x_N \succ y$, and B voters whose preferences are $y \succ x_1 \sim \dots \sim x_N$, and votes are sincere, y will win if and only if $N > \frac{A}{B}$.

Proof: If votes are sincere, then each x_n will receive $\frac{A}{N}$ votes, and y will receive B votes. Therefore, y wins if and only if $B > \frac{A}{N}$, or equivalently, $N > \frac{A}{B}$. Therefore, in this type of situation, having more x candidates is disadvantageous for the $x \succ y$ group. ■

Proposition 10: Given the Borda rule, if the electorate consists of A voters whose preferences are $x_1 \succ \cdots \succ x_N \succ y$, and B voters whose preferences are $y \succ x_1 \succ \cdots \succ x_N$, and votes are sincere, y will win if and only if $N < \frac{B}{A}$.

Proof: y has N candidates ranked above him on A ballots, so his Borda score is $\mathrm{B}_y = NA$. x_1 has one candidate (y) ranked above him on B ballots, so his Borda score is $B_{x_1} = B$. Therefore, y wins if and only if $B > NA$, or equivalently, $N < \frac{B}{A}$. Therefore, in this type of situation, having more x candidates is advantageous for the $x \succ y$ group. ■

Proposition 11: Given the Coombs rule, if the electorate consists of A voters whose preferences are $x_1 \sim \ldots \sim x_N \succ y$, and B voters whose preferences are $y \succ x_1 \sim \ldots \sim x_N$, and votes are sincere, y will win if and only if $N < \frac{B}{A}$.

Proof: This once again follows as a simple calculation. Therefore, in this type of situation, having more x candidates is better for the $x \succ y$ group. ■

Note: Given Hare or minimax, and the scenarios described in propositions 9-11, y wins if and only if $B > A$.

Discussion: Propositions 9 through 11 explore strategic nomination using simple two-group scenarios. They suggest that plurality should be particularly vulnerable to strategic entry, and that Borda and Coombs should be particularly vulnerable to strategic exit. These suggestions are consistent with the simulation results.

9. Conclusion

9.1. Summary of simulation results

Table 1 below summarizes the results presented in sections 6 and 7, by qualitatively characterizing the relative vulnerability of each method to each type of strategic manipulation. (Note that ‘general voting’ refers to the results of analysis V1.)

Table 1: Overall summary

	Hare	runoff	minimax	plurality	approval	range	Coombs	Borda
general voting	very low	low	moderate	moderate	high	high	high	high
compromising	low	low	very low	highest	high	high	none	high
burying	none	none	moderate	none	high	high	high	moderate
exit	moderate	moderate	very low	highest	minimal	minimal	very low	very low
entry	very low	very low	very low	low	minimal	minimal	moderate	highest

9.2. General discussion

With regard to strategic voting, there is a clear stratification between frequently-manipulable methods such as range, Coombs, Borda, and approval, moderately-manipulable methods such as plurality and minimax, and infrequently-manipulable methods such as Hare and runoff: this pattern emerges in all three of the data generating process used here. Plurality is clearly most vulnerable to compromising and strategic exit, while Coombs, range, and approval are most vulnerable to burying, and Borda is most vulnerable to strategic entry.

Among the eight methods that are covered here, Hare has the advantage of being the least frequently vulnerable to strategic voting, but minimax is superior to Hare in resistance to strategic nomination, particularly strategic exit. This result leads one to wonder whether it might be possible to find Condorcet-Hare hybrid methods that possess both of these advantages. Green-Armytage (2011) identifies four methods that appear to fit this description.

Aside from counting the raw frequency with which strategic manipulation is possible, there are many other interesting questions to be explored, such as the likelihood that manipulation will actually occur when it is possible, and the effect of manipulation on social welfare. These questions generally require us to make more assumptions to generate results, but they are nonetheless worth asking, and they have already formed the basis for interesting research.

A broad lesson from this paper is that all voting rules are vulnerable to strategic manipulation in some non-insignificant fraction of elections. Looking at the bottom of table 6, we see that even Hare is vulnerable to strategic voting in a majority of cases if the number of candidates is sufficiently large. Proportional representation may provide a partial solution to this predicament. For example, Tullock (1967) describes a system in which anyone who wishes to can serve as a representative, and in which each representative's voting weight is determined by the number of people who vote for them, whether this number is one, or several million. Since this system would allow all voters to have their first choice of representative, there is arguably no incentive for strategic voting over candidates, though once elected, representatives may still engage in strategic voting over issues, using whatever parliamentary rules they have established. As for the question of which parliamentary rules are least susceptible to manipulation, I leave this for future study.

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Appendix I: Tables

Table 2: Analysis V1, spatial model

V	*S*	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	1	3	.549	.509	.186	.171	.153	.282	.594	.181
99	2	3	.497	.461	.333	.065	.187	.229	.503	.069
99	4	3	.472	.449	.397	.031	.198	.212	.469	.032
99	8	3	.448	.428	.420	.017	.202	.207	.442	.017
99	16	3	.442	.423	.429	.012	.192	.193	.438	.012
99	1	4	.817	.904	.432	.397	.388	.553	.877	.398
99	2	4	.726	.759	.624	.171	.357	.482	.776	.210
99	4	4	.667	.702	.662	.070	.329	.425	.716	.109
99	8	4	.648	.671	.673	.036	.303	.391	.682	.069
99	16	4	.624	.650	.668	.026	.286	.362	.656	.054
99	1	5	.910	.983	.635	.585	.584	.727	.959	.573
99	2	5	.840	.906	.801	.297	.489	.680	.907	.380
99	4	5	.783	.834	.807	.118	.409	.595	.843	.219
99	8	5	.753	.792	.807	.061	.377	.540	.799	.153
99	16	5	.732	.769	.806	.045	.356	.503	.773	.123
99	1	6	.961	.998	.784	.733	.742	.849	.989	.708
99	2	6	.909	.968	.917	.441	.597	.825	.962	.553
99	4	6	.851	.909	.907	.185	.486	.736	.917	.358
99	8	6	.815	.858	.894	.081	.424	.647	.865	.245
99	16	6	.814	.845	.890	.054	.400	.603	.846	.193

Table 3: Analysis V1, impartial culture

V	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
9	3	.599	.586	.623	.136	.352	.389	.606	.136
29	3	.837	.836	.918	.147	.676	.694	.843	.147
99	3	.986	.990	.998	.160	.951	.951	.990	.160
999	3	1.000	1.000	1.000	.166	1.000	1.000	1.000	.166
9	4	.776	.809	.869	.264	.559	.624	.825	.284
29	4	.948	.975	.998	.292	.848	.922	.976	.433
99	4	.999	1.000	1.000	.330	.987	.999	1.000	.667
999	4	1.000	1.000	1.000	.341	1.000	1.000	1.000	.999
9	5	.862	.903	.957	.392	.690	.763	.910	.429
29	5	.978	.995	1.000	.427	.909	.976	.994	.655
99	5	1.000	1.000	1.000	.470	.995	1.000	1.000	.924
999	5	1.000	1.000	1.000	.489	1.000	1.000	1.000	1.000
9	6	.909	.949	.986	.476	.764	.840	.951	.538
29	6	.989	.998	1.000	.541	.945	.992	.998	.803
99	6	1.000	1.000	1.000	.585	.998	1.000	1.000	.984
999	6	1.000	1.000	1.000	.607	1.000	1.000	1.000	1.000

Table 4: Analysis V1, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.578	.553	.657	.031	.280	.289	.587	.030
4	.786	.778	.903	.079	.413	.505	.805	.097
5	.859	.885	.980	.132	.506	.623	.891	.182
6	.882	.921	.985	.221	.568	.696	.928	.276

Table 5: Analysis V1, spatial model, impact of *V*

S	*C*	*V*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
4	3	9	.431	.369	.278	.052	.135	.160	.417	.054
4	3	39	.446	.408	.335	.041	.163	.187	.439	.042
4	3	159	.452	.431	.371	.039	.186	.203	.450	.041
4	3	639	.454	.433	.393	.030	.196	.209	.450	.031
4	3	2559	.468	.447	.405	.029	.197	.208	.467	.030
4	4	9	.465	.443	.402	.028	.201	.214	.461	.028
4	4	39	.464	.442	.410	.027	.204	.216	.457	.027
4	4	159	.476	.451	.412	.028	.213	.226	.472	.029
4	4	639	.472	.452	.413	.029	.209	.220	.470	.029
4	4	2559	.634	.632	.522	.118	.264	.322	.652	.131

Table 6: Analysis V2, compromising and burying, spatial model

V	*S*	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	4	3	.471	.435	.397	.020	.198	.212	.469	.020
99	4	4	.664	.672	.655	.039	.322	.425	.715	.050
99	4	5	.777	.799	.790	.068	.388	.595	.842	.093
99	4	6	.838	.876	.874	.095	.445	.721	.910	.140
99	4	10	.953	.983	.980	.210	.588	.959	.991	.338
99	4	20	.996	1.000	.999	.504	.712	1.000	1.000	.679
99	4	30	1.000	1.000	1.000	.715	.759	1.000	1.000	.822

Table 7: Simple strategic opportunities, as share of all opportunities

	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff	total
spatial	99%	92%	97%	64%	92%	100%	100%	52%	94%
IC	99%	99%	99%	77%	98%	100%	100%	63%	94%
ANES	100%	99%	99%	60%	98%	100%	100%	52%	97%
average	100%	97%	98%	67%	96%	100%	100%	55%	95%

Table 8: Compromising strategy, spatial model

V	*S*	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	4	3	.118	.118	.000	.018	.003	.202	.118	.018
99	4	4	.148	.219	.000	.044	.006	.422	.212	.051
99	4	5	.152	.308	.000	.066	.008	.594	.298	.093
99	4	6	.145	.360	.000	.092	.009	.724	.365	.136

Table 9: Compromising strategy, impartial culture model

V	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
29	3	.451	.493	.000	.118	.084	.700	.481	.118
29	4	.567	.728	.000	.230	.170	.925	.714	.262
29	5	.627	.832	.000	.335	.245	.981	.807	.398
29	6	.657	.880	.000	.407	.304	.992	.869	.480

Table 10: Compromising strategy, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.219	.202	.000	.020	.008	.291	.216	.020
4	.332	.403	.000	.044	.013	.506	.405	.043
5	.455	.557	.000	.073	.023	.627	.545	.082
6	.554	.648	.000	.138	.059	.704	.642	.139

Table 11: Burying strategy, spatial model

V	*S*	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
99	4	3	.374	.294	.382	.000	.148	.000	.374	.000
99	4	4	.605	.459	.634	.000	.235	.000	.639	.000
99	4	5	.735	.549	.784	.000	.288	.000	.787	.000
99	4	6	.810	.593	.859	.000	.303	.000	.869	.000

Table 12: Burying strategy, impartial culture model

V	*C*	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
29	3	.610	.229	.871	.000	.404	.000	.635	.000
29	4	.786	.267	.986	.000	.407	.000	.865	.000
29	5	.865	.293	.998	.000	.418	.000	.937	.000
29	6	.902	.333	.999	.000	.410	.000	.964	.000

Table 13: Burying strategy, ANES

C	approval	Borda	Coombs	Hare	minimax	plurality	range	runoff
3	.495	.297	.647	.000	.209	.000	.448	.000
4	.692	.431	.890	.000	.272	.000	.684	.000
5	.794	.501	.932	.000	.334	.000	.795	.000
6	.827	.538	.979	.000	.328	.000	.873	.000

Table 14: Strategic exit

S	V	C_O	C_I	Borda	Coombs	Hare	minimax	plurality	runoff
4	99	0	3	.006	.001	.015	.001	.091	.015
4	99	0	5	.013	.002	.060	.004	.251	.093
4	99	0	7	.017	.005	.104	.006	.356	.175
4	99	0	9	.021	.007	.151	.008	.434	.267
4	99	0	11	.018	.011	.193	.010	.490	.344
4	99	0	13	.022	.013	.245	.012	.526	.402
4	99	0	15	.026	.016	.298	.013	.546	.448
4	99	0	19	.027	.021	.389	.015	.588	.532
4	99	0	23	.026	.023	.468	.017	.605	.572
4	99	0	27	.022	.031	.533	.018	.627	.597
4	99	0	31	.027	.035	.587	.018	.641	.624
4	99	0	35	.026	.040	.640	.022	.654	.649

Table 15: Strategic entry

S	V	C_I	C_O	Borda	Coombs	Hare	minimax	plurality	runoff
4	99	2	1	.015	.004	.000	.001	.001	.000
4	99	2	2	.029	.006	.000	.001	.004	.000
4	99	2	3	.038	.010	.001	.002	.003	.001
4	99	2	5	.059	.014	.001	.002	.008	.001
4	99	2	7	.065	.018	.002	.002	.009	.002
4	99	2	9	.076	.025	.002	.003	.009	.002
4	99	2	11	.094	.031	.002	.005	.013	.002
4	99	2	13	.103	.036	.002	.005	.016	.002
4	99	2	15	.101	.037	.002	.005	.018	.002
4	99	2	19	.113	.043	.004	.006	.020	.004
4	99	2	23	.118	.050	.004	.008	.020	.004
4	99	2	27	.131	.058	.006	.009	.027	.006
4	99	2	31	.135	.065	.005	.009	.029	.005
4	99	2	35	.139	.071	.004	.010	.030	.004

Appendix 2: Figures

Figure 1: V1, S=4, V=99

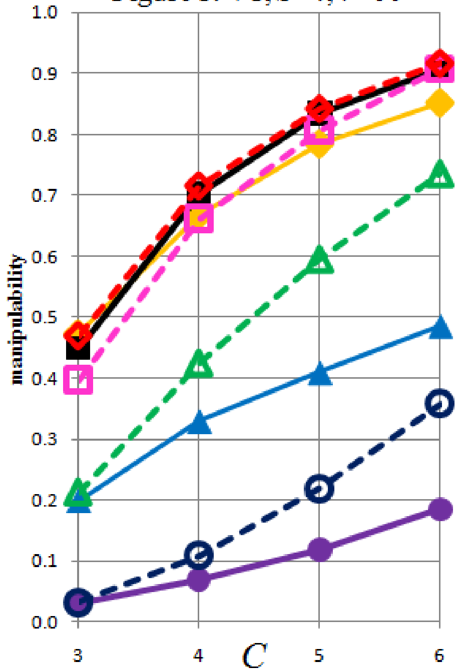


Figure 2: V1, I.C., V=9

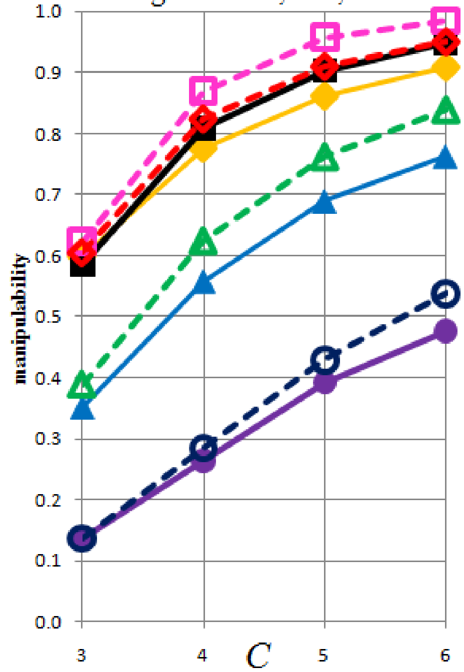
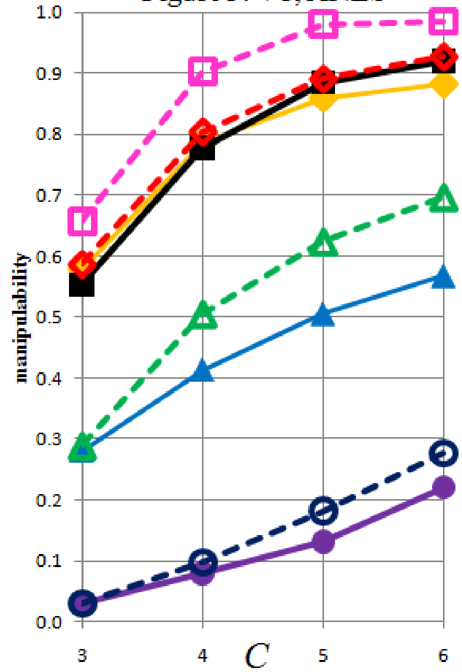


Figure 3: V1, ANES



Hare runoff minimax plurality approval range Borda Coombs

Figure 4: Compromising, S=4, V=99

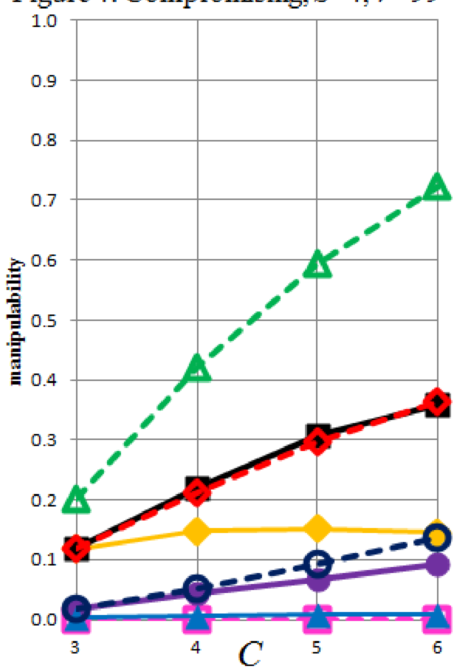


Figure 5: Compromising, I.C., V=29

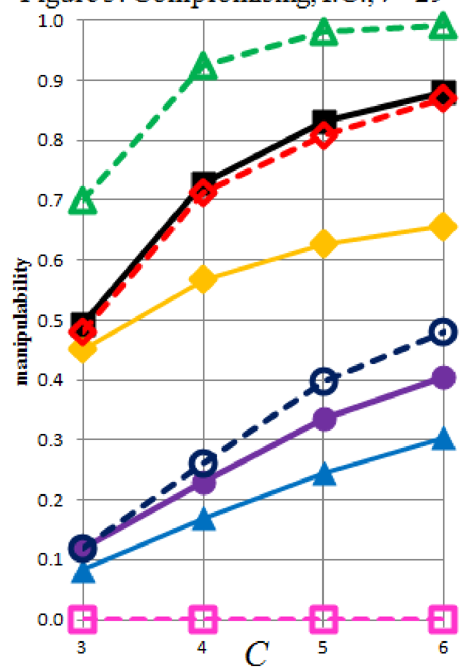
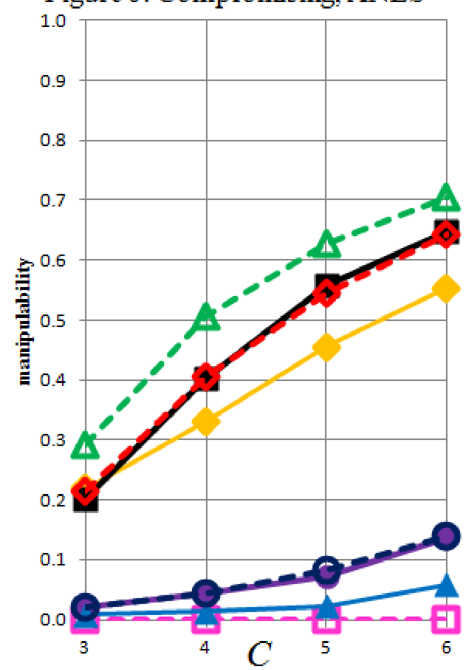


Figure 6: Compromising, ANES



Hare runoff minimax plurality approval range Borda Coombs

Figure 7: Burying, S=4, V=99

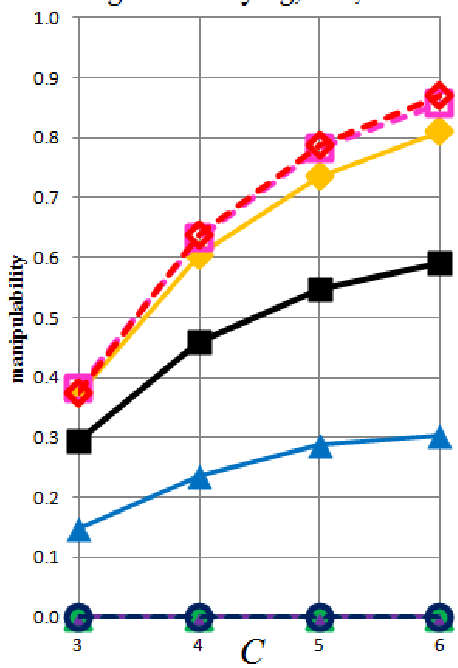


Figure 8: Burying, I.C., V=29

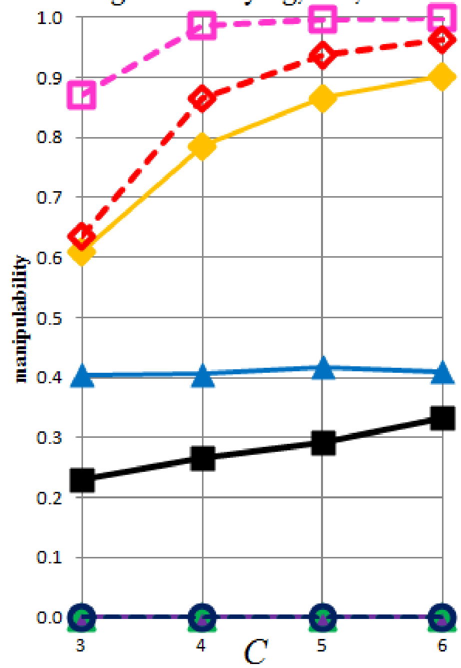


Figure 9: Burying, ANES

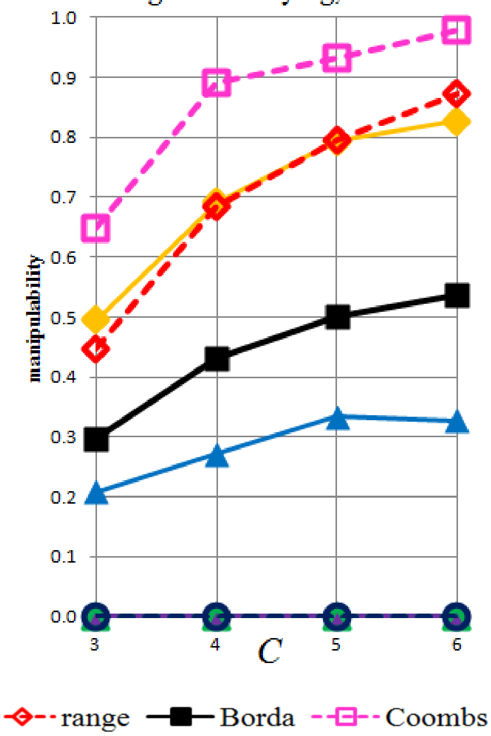


Figure 10: strategic exit, V=99, S=4

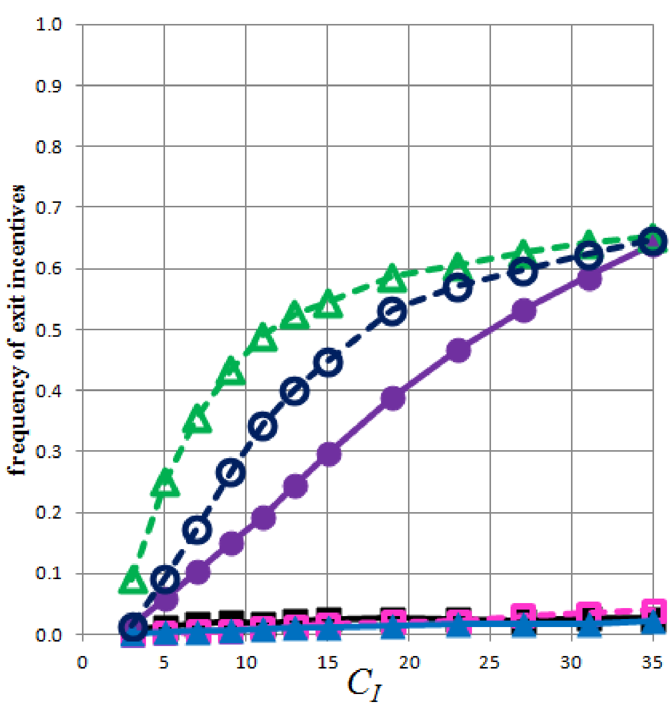


Figure 11: strategic entry, V=99, S=4, C_I=2