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Optimal government size in a simple model

In this simple model, there are two types of markets: private goods markets, which are efficient in the absence of taxes, and public goods markets, in which supply depends entirely on government expenditure. It is assumed that governments can only get revenue to finance this supply by levying per-unit taxes on the private goods markets; thus, the challenge is to optimally balance the efficiency losses associated with this taxation against the efficiency gains associated with providing public goods.

In order to facilitate this balancing act, it will be helpful to think in terms of a common value, $1 + \lambda$, which serves as both the shadow value of government revenue when we are making taxation decisions, and the shadow cost of government expenditure when we are making public goods provision decisions. In part 1, I investigate the optimal level of taxation on a given market as a consequence of λ, and in part 2, I investigate the optimal level of a public good's provision, also as a consequence of λ. In part 3, I generalize the analysis to allow multiple private and public goods, and discuss how questions of optimal taxation and provision can be simultaneously answered by finding the value of λ that leads to the satisfaction of the government's balanced budget constraint. In part 4, I provide an example of this process.

Part 1: Raising revenue in private, initially-efficient markets

Let x be the quantity of a private good, for which there is a competitive, efficient market. Assume that aggregate marginal benefit and marginal cost are given by the following linear functions

$$MB(x) = \alpha - \beta x \qquad\qquad MC(x) = \gamma + \delta x$$

Assume that $\alpha > 0$, $\gamma \geq 0$, $\alpha - \gamma > 0$, $\beta \geq 0$, $\delta \geq 0$, and $\beta + \delta > 0$.

Let τ be a per-unit tax on the good. In the equilibrium with the tax, $MB(x) = MC(x) + \tau$, so

$$\alpha - \beta x = \gamma + \delta x + \tau$$

$$\alpha - \gamma - \tau = x(\beta + \delta)$$

$$x^*(\tau) = \frac{\alpha - \gamma - \tau}{\beta + \delta}$$

$x^*(\tau)$ gives the equilibrium quantity of the good, conditional on τ. Since the market is initially efficient, the surplus-maximizing quantity x^o can be derived by simply setting $\tau = 0$:

$$x^o = \frac{\alpha - \gamma}{\beta + \delta}$$

Let $R(\tau) = \tau x^*(\tau)$ be the government's revenue from the tax, conditional on the tax rate.

$$\boxed{R(\tau) = \frac{\tau(\alpha - \gamma) - \tau^2}{\beta + \delta}}$$

$$R'(\tau) = \frac{\alpha - \gamma - 2\tau}{\beta + \delta}$$

It's important to note that $R(\tau)$ is concave, i.e. that $R''(\tau) < 0$.

Let $DWL(\tau)$ be the deadweight loss (or loss in surplus) as a result of the tax. The formal definition uses an integral, but because supply and demand are linear, deadweight loss can be calculated as the area of a triangle.

$$DWL(\tau) = \int_{x^*}^{x^o} MB(x) - MC(x)\, dx$$

$$DWL(\tau) = \frac{1}{2}\tau(x^o - x^*) = \frac{1}{2}\tau\left(\frac{\tau}{\beta + \delta}\right)$$

$$DWL(\tau) = \frac{\tau^2}{2(\beta + \delta)}$$

$$DWL'(\tau) = \frac{\tau}{\beta + \delta}$$

It's important to note that $DWL(\tau)$ is convex, i.e. that $DWL''(\tau) > 0$. When τ is close to zero, the marginal deadweight loss is close to zero as well, but as τ increases, the ratio of deadweight loss to revenue grows continuously.

Let $1 + \lambda$ be the shadow value of government spending; that is, assume that an additional dollar of public spending (e.g. on public goods) creates $\$1 + \lambda$ in public consumption value. Thus, let $\Omega(\tau)$ be the government's objective function, equal to the difference between the consumption value generated in another (public goods) market as a result of the tax revenue raised in this market, and the consumer and producer surplus lost to this market. The value of the former is $(1 + \lambda)R(\tau)$, and the value of the latter is $R(\tau) + DWL(\tau)$. Thus, the government should choose the value of τ that solves this maximization problem:

$$\max_{\tau} \Omega(\tau) = (1 + \lambda)R(\tau) - [R(\tau) + DWL(\tau)]$$

$$\max_{\tau} \Omega(\tau) = \lambda R(\tau) - DWL(\tau)$$

It's easy to verify that $\Omega''(\tau) < 0$, which guarantees that the maximizing value of τ can be found by setting $\Omega'(\tau) = 0$, and solving for τ, as follows:

$$\Omega'(\tau) = \lambda R'(\tau) - DWL'(\tau) \overset{\text{set}}{=} 0$$

$$\lambda R'(\tau) = DWL'(\tau)$$

$$\lambda \frac{\alpha - \gamma - 2\tau}{\delta + \beta} = \frac{\tau}{\delta + \beta}$$

$$\lambda(\alpha - \gamma) = \tau(2\lambda + 1)$$

$$\boxed{\tau^o = \frac{\lambda(\alpha - \gamma)}{2\lambda + 1}}$$

Thus, $\lambda > 0$ implies $\tau^o > 0$; that is, if $1 + \lambda > 1$, it's optimal to have some positive tax rate despite the fact that this creates deadweight loss in the market for the private good. As one would expect, τ^o depends positively on λ, and positively on $\alpha - \gamma$, i.e. the maximum difference between marginal benefit and marginal cost.

Part 2: Spending money on public goods

So far, we have showed how non-zero taxes on efficient markets can be justified by assuming that government spending has a shadow value $1 + \lambda$ which is greater than one. In order to make this a coherent theory, we should include an explanation of why this might be the case. There are several ways to model the benefits of government spending, but for simplicity I choose to focus on a case in which goods are either purely private or purely public, and in which the public goods are not provided at all in the absence of government expenditure. The quantity of the public good is represented by y, and the parameters of the public goods market mirror those of the private goods market; I only switch from the lower case letters α, β, γ, and δ to the corresponding capital letters A, B, Γ, and Δ, to differentiate between the two.

$$MB(y) = \mathrm{A} - \mathrm{B}y \qquad\qquad MC(y) = \Gamma + \Delta y$$

In the public goods market, our objective is to maximize the total benefit received by those who enjoy the public good, net of government expenditure multiplied by $1 + \lambda$. That is, because we are determining our tax rates with the assumption that $1 + \lambda$ is the shadow value of government spending, the actual per-dollar impact of our public goods provision must be consistent with this. In other words, $1 + \lambda$ must serve dually as the shadow cost of government expenditure, as well as the shadow benefit of government revenue. Thus, defining $TB(y)$ as total benefit, and $E(y)$ as expenditure, our objective function in the public goods market is as follows:

$$\max_{y} \Psi = TB(y) - (1+\lambda)E(y)$$

Setting the first derivative of this function equal to zero, we find the following condition, which can be solved for y in order to obtain the optimal level of expenditure, conditional on λ:

$$MB(y) = (1+\lambda)MC(y)$$

$$\mathrm{A} - \mathrm{B}y = (1+\lambda)(\Gamma + \Delta y)$$

$$\boxed{y^{o} = \frac{\mathrm{A} - (1+\lambda)\Gamma}{\mathrm{B} + (1+\lambda)\Delta}}$$

Given any value of y, we can find the necessary expenditure (total cost) by taking the anti-derivative of the marginal cost function. (We can also find this graphically by adding a rectangle to a right triangle.)

$$\boxed{E(y) = \Gamma y + \frac{1}{2}\Delta y^{2}}$$

Part 3: Global optimality

For the levels of taxation and public goods provision to be optimal, the marginal cost of raising a dollar of revenue $(1+\lambda)$ must be equal to the marginal benefit of spending a dollar of revenue $(1+\lambda)$. This condition applies equally well no matter how many private goods markets there are, and no matter how many public goods there are. Thus, to generalize, I can simply add subscripts $i = 1, \ldots, I$ to parameters and variables pertaining to private goods, and subscripts $j = 1, \ldots, J$ to parameters and variables pertaining to public goods. Thus, the conditions for the optimal per unit tax on private good i, and the resulting revenue, conditional on λ, can be represented as follows:

$$\tau_i^{o}(\lambda) = \frac{\lambda(\alpha_i - \gamma_i)}{2\lambda + 1}$$

$$R_i\big(\tau_i^o(\lambda)\big) = \frac{\tau_i^o(\alpha_i - \gamma_i) - \tau_i^{o\,2}}{\beta_i + \delta_i}$$

Notice that $\partial\tau_i^o(\lambda)/\partial\lambda > 0$, and thus that $\partial R_i\big(\tau_i^o(\lambda)\big)/\partial\lambda > 0$, because the optimal tax rate can't be on the downward-sloping part of the $R_i(\tau_i^o)$ curve. That is, higher values of λ inspire more taxation.

Similarly, the conditions for the optimal provision of public good j, and the resulting expenditure, can be represented as follows:

$$y_j^o(\lambda) = \frac{\mathrm{A_j} - (1+\lambda)\Gamma_\mathrm{j}}{\mathrm{B_j} + (1+\lambda)\Delta_\mathrm{j}}$$

$$E_j\left(y_j^o(\lambda)\right) = \Gamma_\mathrm{j} y_j^o + \frac{1}{2}\Delta_\mathrm{j} y_j^{o\,2}$$

Notice that $\partial y_j^o(\lambda)/\partial\lambda < 0$, and thus that $\partial E_j\left(y_j^o(\lambda)\right)/\partial\lambda < 0$. Thus, as optimal tax revenue is increasing in λ, and as optimal expenditure is decreasing in λ, there must be a unique value λ^o, such that revenue and expenditure are precisely equal.

$$\sum_{i=1}^{I} R_i\big(\tau_i^o(\lambda^o)\big) = \sum_{j=1}^{J} E_j\left(y_j^o(\lambda^o)\right)$$

In the interest of simplicity, we will assume that a balanced budget is necessary. Thus, λ^o is the only value that allows global optimality; once it has been found, the derivation of all policy variables τ_i and y_j follows in a straightforward manner, as described above.

Part 4: Example

In this example, let there be two private goods, let there be two public goods, and let the cost and benefit parameters be as follows:

$$\alpha_1 = 18 \quad \beta_1 = 2 \quad \gamma_1 = 0 \quad \delta_1 = 2$$

$$\alpha_2 = 12 \quad \beta_2 = 2 \quad \gamma_2 = 0 \quad \delta_2 = 2$$

$$\mathrm{A}_1 = 14 \quad \mathrm{B}_1 = 1 \quad \Gamma_1 = 2 \quad \Delta_1 = 0$$

$$\mathrm{A}_2 = 7 \quad \mathrm{B}_2 = 1 \quad \Gamma_2 = 2 \quad \Delta_2 = 0$$

It can be determined (for example, using computational methods) that, given these parameters, the equilibrium value λ^o happens to be 1, so that the shadow value $1 + \lambda^o = 2$. In other words, given optimal taxing and spending, and at the margin, one dollar of government revenue provides two dollars in public goods consumption benefits, and costs two dollars in lost private goods surplus. Rather than focusing on the calculation of λ^o, we will assume that this is given, and work through the calculation of the optimal tax and spending policies, conditional on λ^o. In the end, we will be able to verify that this does in fact lead to a balanced budget.

First, we use the formulas above to calculate the optimal tax rates and resulting revenue:

$$\tau_i^o(\lambda) = \frac{\lambda(\alpha_i - \gamma_i)}{2\lambda + 1} \qquad \tau_1^o(1) = \frac{1(18 - 0)}{2 + 1} \qquad \tau_2^o(1) = \frac{1(12 - 0)}{2 + 1}$$

$$\tau_1^o = 6 \qquad \tau_2^o = 4$$

$$R_i(\tau_i^o) = \frac{\tau_i^o(\alpha_i - \gamma_i) - {\tau_i^o}^2}{\beta_i + \delta_i} \qquad R_1(6) = \frac{6(18 - 0) - 6^2}{2 + 2} \qquad R_2(4) = \frac{4(12 - 0) - 4^2}{2 + 2}$$

$$R_1 = 18 \qquad R_2 = 8$$

Then, we calculate the optimal provision of public goods, and resulting expenditure:

$$y_j^o(\lambda) = \frac{\mathrm{A_j} - (1 + \lambda)\Gamma_\mathrm{j}}{\mathrm{B_j} + (1 + \lambda)\Delta_\mathrm{j}} \qquad y_1^o(1) = \frac{14 - 2 \cdot 2}{1} \qquad y_2^o(1) = \frac{7 - 2 \cdot 2}{1}$$

$$y_1^o = 10 \qquad y_2^o = 3$$

$$E_j\left(y_j^o\right) = \Gamma_\mathrm{j} y_j^o + \frac{1}{2}\Delta_\mathrm{j} {y_j^o}^2 \qquad E_1(10) = 2 \cdot 10 + 0 \qquad E_2(3) = 2 \cdot 3 + 0$$

$$E_1 = 20 \qquad E_2 = 6$$

It is clear now that the budget is balanced, as revenues and expenditures both add to 26. Thus, although we didn't derive $\lambda^{o} = 1$, we have verified that it is correct.

Optimal taxation of the two private goods, and optimal provision of the two public goods, are depicted in the graphs below: